

UNPACKING FRACTIONS

Classroom-Tested Strategies to Build Students'
Mathematical Understanding

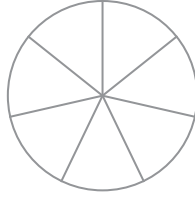
| | |
|--|-----------|
| Foreword by Gail Burrill | xi |
| Introduction: The Challenge of Fractions | 1 |
| Appreciate the Fraction Challenge | 3 |
| From Natural Numbers to Real Numbers | 3 |
| A Word on the Word <i>Fraction</i> | 6 |
| Cognitive Shifts to Consider | 7 |
| The Rush to Algorithms | 10 |
| What Can You Expect from This Book? | 12 |
| Chapter 1: Convey the Many Meanings of $\frac{a}{b}$ | 15 |
| Roberto's Story | 16 |
| Recognizing Misconceptions | 21 |
| Limited Ideas About the Meaning of a Fraction | 21 |
| Difficulty Conceptualizing a Fraction as a Single Number | 22 |
| Unpacking the Mathematical Thinking | 23 |
| The Part-Whole Meaning of $\frac{a}{b}$ | 23 |
| The Measure Meaning of $\frac{a}{b}$ | 27 |
| The Quotient Meaning of $\frac{a}{b}$ | 31 |
| The Ratio Meaning of $\frac{a}{b}$ | 37 |
| The Multiplicative Operator Meaning of $\frac{a}{b}$ | 39 |
| The Rational Number Meaning of $\frac{a}{b}$ | 43 |
| Embodied by the Number Line | 43 |
| Targeting Misconceptions with Challenging Problems | 44 |

| | |
|--|-----------|
| Chapter 2: Use Visual and Tactile Models | 49 |
| Maya's Story | 50 |
| Beyond Misconceptions of Fractions | 51 |
| Limited Repertoire of Fraction Models | 52 |
| Lack of Connectedness Among Models | 53 |
| Unpacking the Mathematical Thinking | 56 |
| Continuous Models | 56 |
| Discrete Models | 60 |
| Discussing and Connecting Models | 61 |
| Targeting Misconceptions with Challenging Problems | 64 |
| Maya's Story, Part 2 | 69 |
| Recognizing Misconceptions | 70 |
| The Parts Need Not Be Equal | 70 |
| The Parts Must Be Clearly Delineated | 70 |
| The Parts Must Have the Same Shape | 70 |
| The Shaded Regions Must Be Grouped into One Part | 71 |
| Unpacking the Mathematical Thinking | 71 |
| The Importance of Equal Parts | 71 |
| Area, Not Shape, Is the Focus | 72 |
| Targeting Misconceptions with Challenging Problems | 73 |
| Maya's Story: Epilogue | 77 |
| Chapter 3: Focus on the Unit | 79 |
| Ed's Story | 80 |
| Recognizing Misconceptions | 83 |
| The Whole Is Made of One Piece | 83 |
| A Fraction Is Smaller Than the Whole, the Unit, or the "1" | 83 |
| Difficulty Conceiving of or Writing Fractions Greater Than 1 | 84 |
| Limited Experience with Non-continuous Units | 84 |
| Unpacking the Mathematical Thinking | 85 |
| The Unit Is Defining | 88 |
| Working with a Variety of Units | 88 |
| Revisiting the Partition and Iteration Process | 90 |
| Targeting Misconceptions with Challenging Problems | 92 |
| Two Vignettes | 99 |
| Linda's Story | 99 |
| Jason's Story | 100 |
| Recognizing Misconceptions | 101 |

| | |
|--|------------|
| Difficulty Going from Part to Whole | 101 |
| Difficulty Discriminating Between What Is Relevant and What Is Not | 101 |
| Unpacking the Mathematical Thinking | 102 |
| A Fraction Is a Relation Between Two Quantities | 102 |
| Proceeding from Part to Whole | 102 |
| Infusing Problems with Distractors: Trapping or Stimulating Students? | 103 |
| Targeting Misconceptions with Challenging Problems | 104 |
| A Final Note | 111 |
| Chapter 4: Teach the Concept of Equivalence (Not Just the Rule) | 112 |
| Lisa's Story | 114 |
| Recognizing Misconceptions | 116 |
| Different Fraction Names for the Same Quantity or Number | 117 |
| Overreliance on Physical Models (3rd Grade and Up) | 117 |
| Difficulty with Discrete Quantities (3rd Grade and Up) | 118 |
| Limited Concept of the Equals Sign (4th Grade and Up) | 119 |
| Rote Application of $\frac{a}{b} = \frac{(n \times a)}{(n \times b)}$ (4th Grade and Up) | 120 |
| The Misuse of Language (All Grades) | 120 |
| A Partial View of the EFA (5th Grade and Up) | 122 |
| Additive Thinking | 123 |
| Unpacking the Mathematical Thinking | 123 |
| Build on Students' Informal Experiences with Equivalence | 124 |
| Cultivate the Equivalence Meaning of <i>Equality</i> | 127 |
| Explain Equivalence by Connecting Fractions to Multiplication and Division | 130 |
| Begin with Equal-Sharing Problem Situations | 131 |
| Model Equivalence Using Different Interpretations of Fractions | 133 |
| Be Mindful That Models Lead to Concept Building | 141 |
| Targeting Misconceptions with Challenging Problems | 142 |
| Chapter 5: Compare and Order Fractions Meaningfully | 148 |
| Nicole's Story | 149 |
| Recognizing Misconceptions | 154 |
| Overreliance on Ready-Made Models | 154 |
| Difficulty Comparing Fractions Without the Common Algorithm | 155 |

| | |
|---|------------|
| Lack of Attention to the Unit | 156 |
| Inappropriate Whole-Number Reasoning | 157 |
| Predominance of Additive Thinking | 158 |
| Unpacking the Mathematical Thinking | 161 |
| Using Models | 162 |
| Reasoning with Unit Fractions | 163 |
| Using the Concept of Equivalence (Common Denominators or Numerators) | 164 |
| Comparing to Benchmarks | 165 |
| Using Multiplicative Thinking | 166 |
| Noticing Patterns | 167 |
| Looking Ahead: Visualizing the “Cross-Product” Method | 170 |
| Targeting Misconceptions with Challenging Problems | 171 |
| Chapter 6: Let Algorithms Emerge Naturally | 177 |
| Vignette 1: Division of a Whole Number by a Fraction | 179 |
| Vignette 2: Multiplication of a Whole Number by a Fraction | 183 |
| Recognizing Misconceptions | 185 |
| Difficulty Seeing Fractions as Numbers | 185 |
| Rote or Incorrect Application of Algorithms | 186 |
| Knowing Fractions Means Knowing the Algorithms | 186 |
| Lack of Fraction Operation Sense | 187 |
| False Beliefs About the Effects of Operations on Numbers or Quantities | 187 |
| Lack of Attention to the Unit | 188 |
| Unpacking the Mathematical Thinking | 188 |
| Begin with Problem Situations That Students Can Tackle | 188 |
| Allow Students to Devise Their Own Algorithms | 190 |
| Revisit Meanings of Addition and Subtraction | 198 |
| Revisit Meanings of Multiplication and Division | 201 |
| Emphasize That Relationships and Properties Still Hold | 208 |
| Highlight Important Changes in Ways of Thinking | 210 |
| Targeting Misconceptions with Challenging Problems | 214 |
| Chapter 7: Connect Fractions and Decimals | 221 |
| Denis’s Story | 223 |
| Recognizing Misconceptions | 227 |
| Scarce Contact with Decimals in Daily Life | 227 |
| Lack of Connectedness Between Fractions and Decimals | 228 |

| | |
|--|------------|
| Difficulty with Symbol Meaning | 229 |
| Overreliance on the Money Model | 230 |
| Poor Understanding of Decimal Magnitude | 231 |
| Rote or Incorrect Application of Decimal Algorithms | 233 |
| Unpacking the Mathematical Thinking | 234 |
| Extending Place Value to Tenths and Hundredths | 234 |
| The Models We Use Are Important | 235 |
| Comparing Decimals Meaningfully | 238 |
| Importance of the Unit | 242 |
| Sensing Approximate Values | 242 |
| Making Sense of Operations | 244 |
| Targeting Misconceptions with Challenging Problems | 248 |
| Conclusion: Moving from Rote to Reason | 255 |
| Foster These Seven Habits of Mind | 255 |
| Teach Meanings First, Algorithms Last | 260 |
| Look Ahead to Ratios, Proportions, Proportional Relations, and Linear Functions | 262 |
| From Fractions to Ratios | 262 |
| From Ratios to Proportions | 265 |
| From Proportions to Proportional Relationships | 266 |
| From Proportional Relationships to Linear Functions | 267 |
| Concluding Thoughts | 269 |
| References | 270 |
| Index | 273 |
| About the Author | 283 |



Introduction

The Challenge of Fractions

The losses that occur because of the gaps in conceptual understanding about fractions, ratios, and related topics are incalculable. The consequences of doing, rather than understanding, directly or indirectly affect a person's attitudes toward mathematics, enjoyment and motivation in learning, course selection in mathematics and science, achievement, career flexibility, and even the ability to fully appreciate some of the simplest phenomena in everyday life.

Susan J. Lamon (2012, p. xi)

The need for better teaching and learning of fractions is one of the few topics in the curriculum with which mathematics educators at every grade level would agree. At conference after conference, teachers bemoan students' resistance to fractions, the trouble they have making sense of them, and their ineptitude at solving problems involving fractions. I offer three fundamental reasons we must take a closer look at how we teach fractions in the United States:

1. *Fractions play a key role in students' feelings about mathematics.*

For many students, fractions present a first mathematical

stumbling block. Students begin disliking mathematics when they must surrender their sense making and yield to sense-less memorization.

2. *Fractions are fundamental to school math and daily life.* Although fractions underpin many complex mathematical topics, including ratios, rates, percents, proportions, proportionality, linearity, and slope, their importance is not limited to mathematical study. As the quote on the previous page indicates, fluency with fractions is also required for many activities of daily life: following recipes, calculating discounts, comparing rates, converting measuring units, reading maps, investing money, and more.

3. *Fractions are foundational to success in algebra.* In its final report, *Foundations for Success*, the National Mathematics Advisory Panel (2008) concluded that (1) algebra is the gateway to success in high school and college, and (2) the main reason for U.S. students' failure in algebra is their poor proficiency with fractions. The worthy goal of "algebra for all" is not possible unless "fractions for all" is a reality. And in our present educational system, a solid grounding in algebra is foundational to a STEM career.

The selection of topics in this book, though by no means exhaustive, was made on the basis of research studies that address the teaching and learning of fractions, evidence from teacher practice, and my own work over the past 25 years with teachers, students, and parents (in both the United States and abroad) from which I have preserved recordings, questions, answers, insights, and samples of student work. It is my hope that readers will find that this book enhances their knowledge of fractions, deepens their appreciation of the complexity involved in teaching them, and perhaps even challenges some long-held beliefs.

Appreciate the Fraction Challenge

No area of elementary mathematics is as mathematically rich, cognitively complicated, and difficult to teach as fractions, ratios, and proportionality.

John P. Smith III (2002, p. 3)

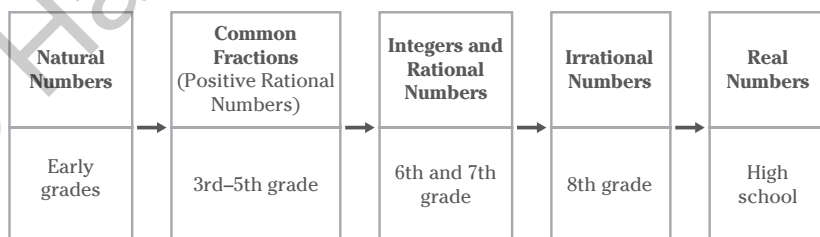
In this section, I introduce the principal reasons fractions are so difficult for students. In Chapters 1 through 7, we'll look at ways to help students move past these difficulties, using strategies and problems that foster understanding of underlying concepts.

From Natural Numbers to Real Numbers

In order to tackle students' greatest challenges with fractions and to feel confident in trying new pedagogical moves, it is important for professionals who work with teachers or students in any grade to know how the number system builds from natural numbers to real numbers (Figure 0.1).

Natural numbers. During early childhood and up to about age 8, children engage with counting numbers, or *natural numbers*—also called *positive whole numbers*. Natural numbers are denoted in mathematics by the symbol \mathbb{N} . In set notation, we write $\mathbb{N} = \{1, 2, 3, 4, \dots\}$. In the United

FIGURE 0.1
Grade-by-Grade Progression from
Natural to Real Numbers



States, 0 is not considered a natural number, but the exclusion of 0 from the set of natural numbers is not universal.

Integers. The next important set of numbers is generated by appending to \mathbb{N} zero and all the “opposites” or “negatives” (*additive inverses*) of the natural numbers. These numbers are called *integers*, denoted by the symbol \mathbb{Z} , and are typically introduced to students in the middle grades. In set notation, we write $\mathbb{Z} = \{ \dots -4, -3, -2, -1, 0, 1, 2, 3, 4 \dots \}$. Since the natural numbers are a subset of the integers, *every natural number is an integer*.

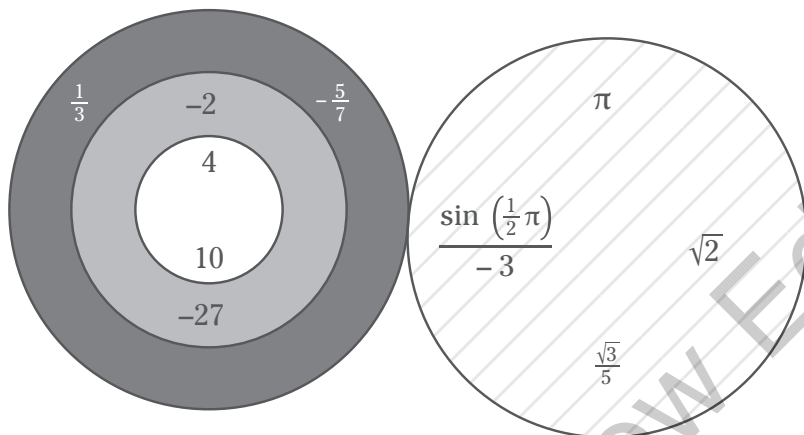
Rational numbers. The common fractions introduced in the upper elementary school grades, such as $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{2}{3}$, are a subset of the *rational numbers*. Rational numbers, denoted by the symbol \mathbb{Q} , are numbers that can be expressed in the form $\frac{a}{b}$, where a and b are integers, provided $b \neq 0$. In set notation, we write $\mathbb{Q} = \left\{ \frac{a}{b}, \text{ where } a \text{ and } b \text{ are members of } \mathbb{Z}, \text{ but } b \neq 0 \right\}$. Every rational number has an equivalent decimal form—for instance, $\frac{1}{2} = 0.5$. Thinking of the symbol $\frac{a}{b}$ as a *quotient* of two integers helps students remember the symbol \mathbb{Q} .

Notice that any integer can be written in the form $\frac{a}{b}$ in many ways. For example, -5 can be written as $\frac{-5}{1}$ or $-\frac{10}{2}$, and $+7$ can be written as $\frac{+7}{1}$ or $\frac{21}{3}$. Therefore, *every integer is a rational number*.

Irrational numbers. In 7th or 8th grade, students learn about a whole new set of numbers, such as π or $\sqrt{2}$, which cannot be written as quotients of two integers. These are known as *irrational numbers*, because they didn't make sense to the ancient Greeks who discovered them. Irrational numbers do not have a universally accepted symbol, although I is often used. Unlike the two preceding relationships, the rational numbers are not a subset of the irrational numbers; rather, the two sets are mutually exclusive.

Real numbers. Rational and irrational numbers together form the set of *real numbers*, denoted by the letter \mathbb{R} . By high school, the universe of numbers within which students operate has grown to include all real numbers as shown in Figure 0.2 on the next page.

FIGURE 0.2
The Real Number System



Note: The real number system contains \mathbb{N} , the natural numbers symbolized by the white set; \mathbb{Z} , the integers symbolized by the light gray set; \mathbb{Q} , the rational numbers symbolized by the dark gray set; and I , the irrational numbers (whose symbol is not universal), symbolized by the hatched set.

At this point, you may be wondering, “Aren’t rational numbers a *middle school mathematics topic*?” Yes, but not exclusively. The higher expectations of the K–12 Common Core State Standards for Mathematics (CCSSM) require a more profound exposure to rational numbers before the middle grades. In fact, the CCSSM formally introduce fractions in 3rd grade, building on students’ prior informal experiences, such as cutting apples into equal halves or sharing a chocolate bar fairly among four people. A recent National Council of Teachers of Mathematics (NCTM) publication for teachers explicitly states, “Rational numbers compose a major area of school mathematics that is crucial for students to learn but challenging for teachers to teach. Students in grades 3–5 need to understand rational numbers well if they are to succeed in these grades and in their subsequent mathematics experiences” (Barnett-Clarke, Fisher, Marks, & Ross, 2010, p. 1).