UNPACKING FRACTIONS

Classroom-Tested Strategies to Build Students' Mathematical Understanding

Introduction

The Challenge of Fractions

The losses that occur because of the gaps in conceptual understanding about fractions, ratios, and related topics are incalculable. The consequences of doing, rather than understanding, directly or indirectly affect a person's attitudes toward mathematics, enjoyment and motivation in learning, course selection in mathematics and science, achievement, career flexibility, and even the ability to fully appreciate some of the simplest phenomena in everyday life.

Susan J. Lamon (2012, p. xi)

The need for better teaching and learning of fractions is one of the few topics in the curriculum with which mathematics educators at every grade level would agree. At conference after conference, teachers bemoan students' resistance to fractions, the trouble they have making sense of them, and their ineptitude at solving problems involving fractions. I offer three fundamental reasons we must take a closer look at how we teach fractions in the United States:

1. *Fractions play a key role in students' feelings about mathematics*. For many students, fractions present a first mathematical

stumbling block. Students begin disliking mathematics when they must surrender their sense making and yield to sense-*less* memorization.

2. *Fractions are fundamental to school math and daily life*. Although fractions underpin many complex mathematical topics, including ratios, rates, percents, proportions, proportionality, linearity, and slope, their importance is not limited to mathematical study. As the quote on the previous page indicates, fluency with fractions is also required for many activities of daily life: following recipes, calculating discounts, comparing rates, converting measuring units, reading maps, investing money, and more.
3. *Fractions are foundational to success in algebra*. In its final report, *Foundations for Success*, the National Mathematics Advisory Panel (2008) concluded that (1) algebra is the gateway to success in high school and college, and (2) the main reason for U.S. students' failure in algebra is their poor proficiency with fractions. The worthy goal of "algebra for all" is not possible unless "fractions for all" is a reality. And in our present educational system, a solid grounding in algebra is foundational to a STEM career.

The selection of topics in this book, though by no means exhaustive, was made on the basis of research studies that address the teaching and learning of fractions, evidence from teacher practice, and my own work over the past 25 years with teachers, students, and parents (in both the United States and abroad) from which I have preserved recordings, questions, answers, insights, and samples of student work. It is my hope that readers will find that this book enhances their knowledge of fractions, deepens their appreciation of the complexity involved in teaching them, and perhaps even challenges some long-held beliefs.

Appreciate the Fraction Challenge

No area of elementary mathematics is as mathematically rich, cognitively complicated, and difficult to teach as fractions, ratios, and proportionality.
John P. Smith III (2002, p. 3)

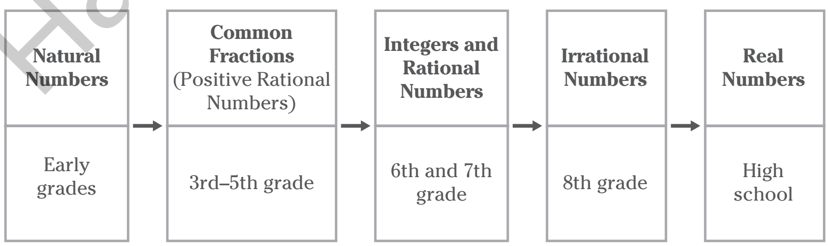
In this section, I introduce the principal reasons fractions are so difficult for students. In Chapters 1 through 7, we'll look at ways to help students move past these difficulties, using strategies and problems that foster understanding of underlying concepts.

From Natural Numbers to Real Numbers

In order to tackle students' greatest challenges with fractions and to feel confident in trying new pedagogical moves, it is important for professionals who work with teachers or students in any grade to know how the number system builds from natural numbers to real numbers (Figure 0.1).

Natural numbers. During early childhood and up to about age 8, children engage with counting numbers, or *natural numbers*—also called *positive whole numbers*. Natural numbers are denoted in mathematics by the symbol $\mathbb{N}$. In set notation, we write $\mathbb{N} = \{1, 2, 3, 4. . .\}$. In the United

FIGURE 0.1
Grade-by-Grade Progression from Natural to Real Numbers



States, 0 is not considered a natural number, but the exclusion of 0 from the set of natural numbers is not universal.

Integers. The next important set of numbers is generated by appending to $\mathbb{N}$ zero and all the "opposites" or "negatives" (*additive inverses*) of the natural numbers. These numbers are called *integers*, denoted by the symbol $\mathbb{Z}$, and are typically introduced to students in the middle grades. In set notation, we write $\mathbb{Z} = \{\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots\}$. Since the natural numbers are a subset of the integers, *every natural number is an integer*.

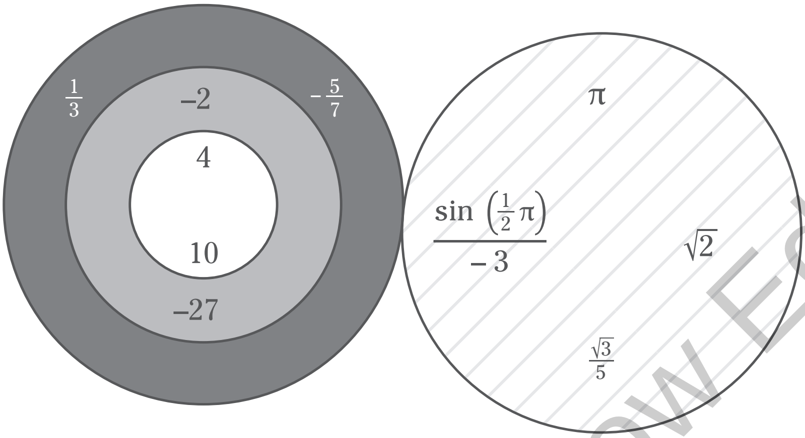
Rational numbers. The common fractions introduced in the upper elementary school grades, such as $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{2}{3}$, are a subset of the *rational numbers*. Rational numbers, denoted by the symbol $\mathbb{Q}$, are numbers that can be expressed in the form $\frac{a}{b}$, where a and b are integers, provided $b \neq 0$. In set notation, we write $\mathbb{Q} = \left\{\frac{a}{b}, \text{ where } a \text{ and } b \text{ are members of } \mathbb{Z}, \text{ but } b \neq 0\right\}$. Every rational number has an equivalent decimal form—for instance, $\frac{1}{2} = 0.5$. Thinking of the symbol $\frac{a}{b}$ as a *quotient* of two integers helps students remember the symbol $\mathbb{Q}$.

Notice that any integer can be written in the form $\frac{a}{b}$ in many ways. For example, -5 can be written as $\frac{-5}{1}$ or $-\frac{10}{2}$, and $+7$ can be written as $\frac{+7}{1}$ or $\frac{21}{3}$. Therefore, *every integer is a rational number*.

Irrational numbers. In 7th or 8th grade, students learn about a whole new set of numbers, such as π or $\sqrt{2}$, which cannot be written as quotients of two integers. These are known as *irrational numbers*, because they didn't make sense to the ancient Greeks who discovered them. Irrational numbers do not have a universally accepted symbol, although *I* is often used. Unlike the two preceding relationships, the rational numbers are not a subset of the irrational numbers; rather, the two sets are mutually exclusive.

Real numbers. Rational and irrational numbers together form the set of *real numbers*, denoted by the letter $\mathbb{R}$. By high school, the universe of numbers within which students operate has grown to include all real numbers as shown in Figure 0.2 on the next page.

FIGURE 0.2
The Real Number System



Note: The real number system contains ℕ, the natural numbers symbolized by the white set; ℤ, the integers symbolized by the light gray set; ℚ, the rational numbers symbolized by the dark gray set; and *I*, the irrational numbers (whose symbol is not universal), symbolized by the hatched set.

At this point, you may be wondering, "Aren't rational numbers a *middle school mathematics topic*?" Yes, but not exclusively. The higher expectations of the K–12 Common Core State Standards for Mathematics (CCSSM) require a more profound exposure to rational numbers before the middle grades. In fact, the CCSSM formally introduce fractions in 3rd grade, building on students' prior informal experiences, such as cutting apples into equal halves or sharing a chocolate bar fairly among four people. A recent National Council of Teachers of Mathematics (NCTM) publication for teachers explicitly states, "Rational numbers compose a major area of school mathematics that is crucial for students to learn but challenging for teachers to teach. Students in grades 3–5 need to understand rational numbers well if they are to succeed in these grades and in their subsequent mathematics experiences" (Barnett-Clarke, Fisher, Marks, & Ross, 2010, p. 1).