
Contents

| | |
|----------------------------------------------------------------------------------------------|-----------|
| Preface | ix |
| About the Authors | xi |
| Introductory Idea | 1 |
| Coming to Terms With Mathematical Terms | 3 |
| Algebra Ideas | 7 |
| 1. Introducing the Product of Two Negatives | 9 |
| 2. Multiplying Polynomials by Monomials (Introducing Algebra Tiles) | 11 |
| 3. Multiplying Binomials (Using Algebra Tiles) | 13 |
| 4. Factoring Trinomials (Using Algebra Tiles) | 16 |
| 5. Multiplying Binomials (Geometrically) | 20 |
| 6. Factoring Trinomials (Geometrically) | 22 |
| 7. Trinomial Factoring | 24 |
| 8. How Algebra Can Be Helpful | 25 |
| 9. Automatic Factoring of a Trinomial | 26 |
| 10. Reasoning Through Algebra | 28 |
| 11. Pattern Recognition Cautions | 29 |
| 12. Caution With Patterns | 30 |
| 13. Using a Parabola as a Calculator | 32 |
| 14. Introducing Literal Equations: Simple Algebra to Investigate an Arithmetic Phenomenon | 35 |
| 15. Introducing Nonpositive Integer Exponents | 39 |
| 16. Importance of Definitions in Mathematics (Algebra) | 41 |
| 17. Introduction to Functions | 44 |
| 18. When Algebra Explains Arithmetic | 45 |
| 19. Sum of an Arithmetic Progression | 46 |
| 20. Averaging Rates | 48 |
| 21. Using Triangular Numbers to Generate Interesting Relationships | 50 |
| 22. Introducing the Solution of Quadratic Equations Through Factoring | 52 |
| 23. Rationalizing the Denominator | 53 |
| 24. Paper Folding to Generate a Parabola | 54 |

| | |
|--------------------------------------------------------------------------------|-----------|
| 25. Paper Folding to Generate an Ellipse | 55 |
| 26. Paper Folding to Generate a Hyperbola | 57 |
| 27. Using Concentric Circles to Generate a Parabola | 59 |
| 28. Using Concentric Circles to Generate an Ellipse | 60 |
| 29. Using Concentric Circles to Generate a Hyperbola | 61 |
| 30. Summing a Series of Powers | 62 |
| 31. Sum of Limits | 65 |
| 32. Linear Equations With Two Variables | 67 |
| 33. Introducing Compound Interest Using the “Rule of 72” | 70 |
| 34. Generating Pythagorean Triples | 72 |
| 35. Finding Sums of Finite Series Geometry Ideas | 74 |
| Geometry Ideas | 79 |
| 1. Sum of the Measures of the Angles of a Triangle | 81 |
| 2. Introducing the Sum of the Measures of the Interior Angles of a Polygon | 82 |
| 3. Sum of the Measures of the Exterior Angles of a Polygon: I | 84 |
| 4. Sum of the Measures of the Exterior Angles of a Polygon: II | 86 |
| 5. Triangle Inequality | 88 |
| 6. Don’t Necessarily Trust Your Geometric Intuition | 89 |
| 7. Importance of Definitions in Mathematics (Geometry) | 93 |
| 8. Proving Quadrilaterals to Be Parallelograms | 97 |
| 9. Demonstrating the Need to Consider All Information Given | 98 |
| 10. Midlines of a Triangle | 100 |
| 11. Length of the Median of a Trapezoid | 102 |
| 12. Pythagorean Theorem | 104 |
| 13. Simple Proofs of the Pythagorean Theorem | 107 |
| 14. Angle Measurement With a Circle by Moving the Circle | 110 |
| 15. Angle Measurement With a Circle | 114 |
| 16. Introducing and Motivating the Measure of an Angle Formed by Two Chords | 116 |
| 17. Using the Property of the Opposite Angles of an Inscribed Quadrilateral | 118 |
| 18. Introducing the Concept of Slope | 119 |
| 19. Introducing Concurrency Through Paper Folding | 122 |
| 20. Introducing the Centroid of a Triangle | 124 |
| 21. Introducing the Centroid of a Triangle Via a Property | 125 |
| 22. Introducing Regular Polygons | 127 |
| 23. Introducing π | 129 |
| 24. The Lunes and the Triangle | 132 |
| 25. The Area of a Circle | 135 |
| 26. Comparing Areas of Similar Polygons | 136 |
| 27. Relating Circles | 138 |
| 28. Invariants in Geometry | 139 |
| 29. Dynamic Geometry to Find an Optimum Situation | 142 |
| 30. Construction-Restricted Circles | 144 |
| 31. Avoiding Mistakes in Geometric Proofs | 146 |

| | |
|---------------------------------------------------------------------------------------------------|------------|
| 32. Systematic Order in Successive Geometric Moves: Patterns! | 150 |
| 33. Introducing the Construction of a Regular Pentagon | 153 |
| 34. Euclidean Constructions and the Parabola | 158 |
| 35. Euclidean Constructions and the Ellipse | 165 |
| 36. Euclidean Constructions and the Hyperbola | 170 |
| 37. Constructing Tangents to a Parabola From an External Point P | 177 |
| 38. Constructing Tangents to an Ellipse | 180 |
| 39. Constructing Tangents to a Hyperbola | 182 |
| Trigonometry Ideas | 187 |
| 1. Derivation of the Law of Sines: I | 189 |
| 2. Derivation of the Law of Sines: II | 190 |
| 3. Derivation of the Law of Sines: III | 191 |
| 4. A Simple Derivation for the Sine of the Sum of Two Angles | 192 |
| 5. Introductory Excursion to Enable an Alternate Approach to Trigonometry Relationships | 194 |
| 6. Using Ptolemy's Theorem to Develop Trigonometric Identities for Sums and Differences of Angles | 197 |
| 7. Introducing the Law of Cosines: I (Using Ptolemy's Theorem) | 202 |
| 8. Introducing the Law of Cosines: II | 204 |
| 9. Introducing the Law of Cosines: III | 206 |
| 10. Alternate Approach to Introducing Trigonometric Identities | 207 |
| 11. Converting to Sines and Cosines | 210 |
| 12. Using the Double Angle Formula for the Sine Function | 211 |
| 13. Making the Angle Sum Function Meaningful | 212 |
| 14. Responding to the Angle-Trisection Question | 214 |
| Probability and Statistics Ideas | 217 |
| 1. Introduction of a Sample Space | 219 |
| 2. Using Sample Spaces to Solve Tricky Probability Problems | 220 |
| 3. Introducing Probability Through Counting (or Probability as Relative Frequency) | 222 |
| 4. In Probability You Cannot Always Rely on Your Intuition | 223 |
| 5. When "Averages" Are Not Averages: Introducing Weighted Averages | 225 |
| 6. The Monty Hall Problem: "Let's Make a Deal" | 226 |
| 7. Conditional Probability in Geometry | 229 |
| 8. Introducing the Pascal Triangle | 230 |
| 9. Comparing Means Algebraically | 233 |
| 10. Comparing Means Geometrically | 234 |
| 11. Gambling Can Be Deceptive | 236 |
| Other Topics Ideas | 237 |
| 1. Asking the Right Questions | 239 |
| 2. Making Arithmetic Means Meaningful | 240 |
| 3. Using Place Value to Strengthen Reasoning Ability | 241 |

| | |
|--------------------------------------------------------------------------------|-----|
| 4. Prime Numbers | 242 |
| 5. Introducing the Concept of Relativity | 245 |
| 6. Introduction to Number Theory | 246 |
| 7. Extracting a Square Root | 247 |
| 8. Introducing Indirect Proof | 248 |
| 9. Keeping Differentiation Meaningful | 248 |
| 10. Irrationality of \sqrt{m} | 250 |
| 11. Introduction to the Factorial Function $x!$ | 252 |
| 12. Introduction to the Function $x^{(n)}$ | 254 |
| 13. Introduction to the Two Binomial Theorems | 257 |
| 14. Factorial Function Revisited | 259 |
| 15. Extension of the Factorial Function $r!$ to the Case Where r Is Rational | 262 |
| 16. Prime Numbers Revisited | 276 |
| 17. Perfect Numbers | 278 |

© Hawker Brownlow Education

Introducing the Product of Two Negatives

Objective: To have students understand intuitively and abstractly why the product of two negatives is positive

Materials: A video camera and a videotape player

Procedure: Typically, a discussion of the product of two negative numbers follows a discussion of the product of a negative number and a positive number (which may be shown as the multiple addition of a negative number or by some other convenient method). This product generally poses no great difficulty for the teacher to demonstrate or for the students to understand.

It is more difficult to develop an analogue for the product of two negative numbers. We can show that the product of two negatives evolves from the pattern:

$$\begin{aligned}(-1)(+3) &= -3 \\(-1)(+2) &= -2 \\(-1)(+1) &= -1 \\(-1)(0) &= 0\end{aligned}$$

Continuing the pattern seen by successive entries in the second and third numbers of each line gives the following:

$$\begin{aligned}(-1)(-1) &= +1 \\(-1)(-2) &= +2 \\(-1)(-3) &= +3\end{aligned}$$

How can students get a more genuine intuitive feel for this concept? Perhaps a “real-life” illustration will do.

Consider making a videotape recording of a clear plastic water tank that has a transparent drain tube that we know can empty (*negative*) the tank at the rate of 3 gallons per minute. We tape this event for several minutes, allowing the tank contents to lower. If we run the tape for 2 minutes, the tank will show a decrease of 6 gallons. For 3 minutes, the tank will have emptied 9 gallons.

10 Algebra Ideas

Suppose we run the tape in reverse (*negative*) for 1 minute. The tank will have refilled by 3 gallons. At twice the normal rate in reverse (*negative*), the tank will gain (*positive*) in content at the rate of 6 gallons per minute. Here a student can see that the product of two negatives (emptying a tank and running a film in reverse) results in a positive.

If this is not sufficiently convincing, then perhaps tell students to consider **good guys** (positive) or **bad guys** (negative) who are **entering** (positive) or **leaving** (negative) a town:

1. If the **good guys** (+) **enter** (+) the town, that is **good** (+) for the town. $(+)(+) = +$
2. If the **good guys** (+) **leave** (-) the town, that is **bad** (-) for the town. $(+)(-) = -$
3. If the **bad guys** (-) **enter** (+) the town, that is **bad** (-) for the town. $(-)(+) = -$
4. If the **bad guys** (-) **leave** (-) the town, that is **good** (+) for the town. $(-)(-) = +$

This example gives students an intuitive feel for the product of two negatives. You, the teacher, should choose the approach most likely to succeed with a particular class.

For the more curious or gifted students, a proof of this concept might be in order. They should know that $(-1) + 1 = 0$. By multiplying both sides of this equation by -1 , we get

$$(-1)[(-1) + 1] = (-1)[0]$$

Using the distributive property, we get

$$(-1)(-1) + (-1)1 = (-1)0$$

Because we know that

$$(-1)1 = -1 \text{ and } (-1)0 = 0$$

we then have

$$(-1)(-1) + (-1) = 0$$

By adding 1 to both sides of this equation (or by asking, "What must be added to (-1) to get 0?"), we get

$$(-1)(-1) = 1$$

which "proves" the relationship.

We might also like to reach back to another topic to validate further that the product of two negatives is a positive. Recall that $(a^x)^y = a^{xy}$, which holds true for $a = 2$, $x = y = -1$.

Therefore,

$$(2^{-1})^{-1} = 2^{(-1)(-1)}$$

By the definition of a negative exponent, we get

$$(2^{-1})^{-1} = \left(\frac{1}{2}\right)^{-1} = 2$$

We can then conclude that

$$2^1 = 2^{(-1)(-1)} \text{ or } 1 = (-1)(-1)$$

because if the bases are equal, so must the exponents be equal.

ALGEBRA IDEA 2

Multiplying Polynomials by Monomials (Introducing Algebra Tiles)

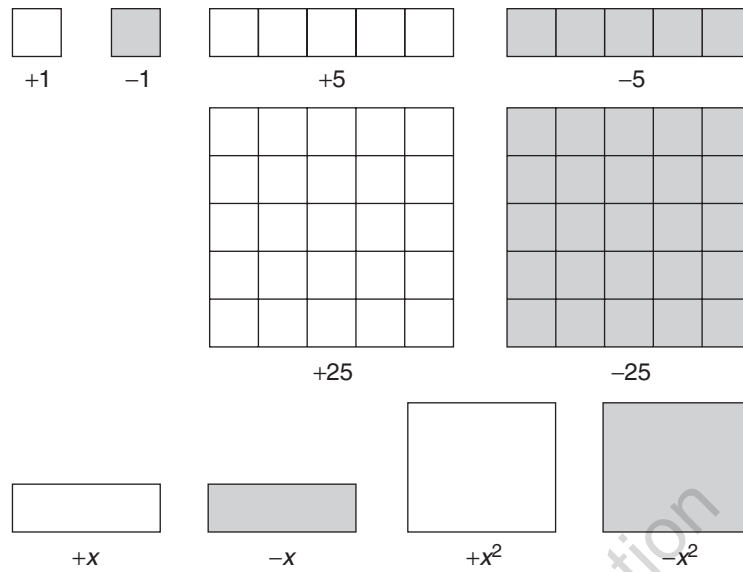
Objective: To examine the multiplication of polynomials by monomials geometrically

Materials: Algebra tiles

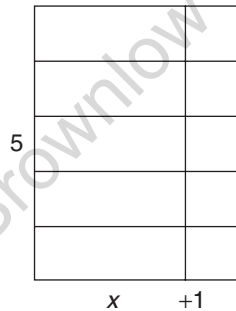
Procedure: In recent years, the use of algebra tiles has enhanced the teaching of abstract concepts and has helped to make working with polynomials more concrete. Depending on the abilities of your students, you may need an entire lesson to introduce the different algebra tiles and to familiarize your students with them. There are commercially made algebra tiles; however, you can make your own by duplicating the following diagrams on colored paper and cutting them out. Each student should have 10 of each type of tile. Here the nonshaded tiles represent positive numbers and the shaded tiles represent negative numbers.

The number associated with a rectangular tile represents its area and color. For example, the +5 tile has dimensions 5 by 1, and thus its area is 5. Where it is unshaded (clear), we use the number +5 to identify it. The -5 tile has a similar situation, except that it is shaded, and therefore we use the number -5 to identify it. Likewise, the + x tile has dimensions x by 1, and its area is, therefore, x . Note that because we are *not* given the length of this tile, we denote it with an x . Note also that an integral number of units do not fit the length.

12 Algebra Ideas



Pose the question, “How can we multiply $(x + 1)$ by 5 using algebra tiles?” If a rectangle is to be created with area $(x + 1) \cdot 5$, it should have length $(x + 1)$ and width 5. Have the students build such a rectangle with their tiles as follows:

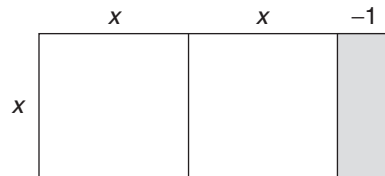


After building such a rectangle, have the students count the algebra tiles to find that the area associated with the rectangle is $5x + 5$. This shows that

$$(x + 1) \cdot 5 = 5x + 5$$

This is equivalent to the distributive law.

Now consider the product $(2x - 1) \cdot x$. The resulting rectangle is



Note that the tile on the right must be a $-x$ tile because the -1 is one side length. Adding the results represented by the three tiles gives

$$x^2 + x^2 + (-x) \text{ or } 2x^2 - x$$

It may be appropriate to show how the distributive law applies here as well.

Before expanding the demonstration to the product of binomials, have the students practice using the technique, especially with respect to signed numbers. Have students use algebra tiles to multiply

1. $(x - 3) \cdot 4$
2. $(2x - 2) \cdot (-1)$
3. $(3 - 2x) \cdot (3)$
4. $(-x + 5) \cdot (-5)$
5. $(x^2 - 3) \cdot (-4)$
6. $(x + 2x - 3) \cdot (3)$

Ask students which extra tiles are needed to consider products that involve the variables x and y .

ALGEBRA IDEA 3

Multiplying Binomials (Using Algebra Tiles)

Objective: To examine the multiplication of binomials by binomials geometrically

Materials: Algebra tiles

Procedure: The basis for this unit is to use the previous work on algebra tiles (see Algebra Idea 2) to multiply binomials. Students should be able to represent algebraically the area of a rectangle, whose sides represent binomials.

Consider the product $(x + 3)^2$. Some students may incorrectly give the answer $x^2 + 9$. To see the correct answer, suggest that students use their algebra tiles to construct a square, each of whose sides has length $x + 3$. The resulting square should look like