Contents

INTRODUCTION TO THE ACADEMIC RESEARCH

This book provides a model for diagnosing errors in computation and providing *meaningful instructional strategies for timely, pinpointed intervention.* The book begins with a two-part section called "Big Ideas in Computation and Problem Solving." That section is included because before students consider specific algorithms, they should have an understanding of the role our base-ten place-value system plays in multidigit computation – along with the types of *actions* and *problem structures* that are suggested by each operation.

Each unit on computation begins with a diagnostic test (in multiple-choice format), followed by an Item Analysis Table that keys student incorrect test responses to specific error patterns. Each distractor on the tests is based on a specific error pattern. A comprehensive section, "Error Patterns & Intervention Activities," then follows. This section provides detailed analysis of error patterns with supporting Intervention Activities for each operation. The items used on the diagnostic tests are drawn from this section. Each unit ends with a short section of supplemental practice.

Lack of Conceptual Understanding – Error Patterns

"[Children frequently] either fail to grasp the concepts that underlie procedures or cannot connect the concepts to the procedures. Either way, children who lack such understanding frequently generate flawed procedures that result in systematic patterns of errors. . . . The errors are an opportunity in that their systematic quality points to the source of the problem and this indicates the specific misunderstanding that needs to be overcome."

– Siegler (2003, p. 291)

Beattie and Algozzine (1982) note that when teachers use diagnostic tests to look for error patterns, "testing for teaching begins to evolve" (p. 47). And because diagnostic testing is just one of many tools to analyse student understanding, with each Item Analysis Table are additional suggestions to delve into the rationale of student errors.

According to Thanheiser (2009), "To help their students learn about numbers and algorithms, teachers need more than ability to perform algorithms. They need to be able to explain the mathematics underlying the algorithms in a way that will help children understand" (p. 277). Research by Hill, Rowan and Ball (2005) found that this type of knowledge, known as mathematical knowledge for teaching (or pedagogical content knowledge), positively predicted mathematics student achievement gains in years one and three.

Linking Research and Practice

"The call for a better linking of research and practice has been echoed in the mathematics education community for some time."

– Arbaugh et al. (2010, p. 4)

The Intervention Activities in this book are based on instructional practices supported by academic research that teach for *meaning.* The activities place a strong emphasis on using *place value* as a way to develop this understanding. The practices employed include activating prior knowledge, using representations, using estimation and mental maths, introducing alternative algorithms and participating in instructional games.

According to Kilpatrick, Swafford and Bradford (2001), "when students practice procedures they do not understand, there is a danger they will practice incorrect procedures, thereby making it more difficult to learn correct ones. … Further, when students learn a procedure without understanding, they need extensive practice so as not to forget the steps" (pp. 122–123).

A common subtraction error is shown on the right. Fuson and Briars (1990) found that students who learn to subtract with *understanding* rarely make this error.

A Common Subtraction Error Pattern

$$\begin{array}{r} 92 \\ -28 \\ \hline 76 \end{array}$$

The student subtracts the lesser digit from the greater digit in each place-value position, ignoring order (and renaming)..

An important premise of this book is that when teachers analyse student work for conceptual and procedural misconceptions – and then provide timely, targeted and meaningful intervention – the probability of the errors repeating in the future decreases. Hill, Ball and Schilling (2008), citing the research of others, found when teachers investigated how students learn particular subject matter, such as whole-number operations, "their classroom practices changed and student learning was improved over that of teachers in comparison groups" (p. 376). According to Cox (1975), systematic errors (errors that occur in at least three out of five problems for a specific algorithmic computation) are potentially remediable, "but without proper instructional intervention the systematic errors will continue for long periods of time" (p. 152).

"As we teach computation procedures, we need to remember that our students are not necessarily learning what we think we are teaching; we need to keep our eyes and ears open to find out what our students are actually learning. We need to be alert for error patterns!"

– Ashlock (2010, p. 14)

It is important to emphasise that class or individual discussions of the errors should be conducted as part of a *positive* learning experience – one that allows for students to use reasoning and problem solving to explore why an erroneous procedure may not yield the correct answer.

Finally, any discussion on intervention would be incomplete without addressing key factors that affect the entire child, such as the principle of equity, student dispositions and differentiating instruction. These areas are addressed in this research chapter.

EQUITY AND QUALITY IN THE MATHS CLASSROOM

Equity and *quality* in the maths classroom often imply providing every student with both an equal and a quality learning experience. Hiebert and colleagues (1997) define equity such that "every learner – bilingual students, handicapped students, students of all ethnic groups, students who live in poverty, girls, and boys – can learn mathematics with understanding. In order to do this, *each* student must have access to learning with understanding" (p. 65).

The Equity Principle

"Excellence in mathematics education requires equity – high expectations and strong support for all students."

– NCTM (2000, p. 12)

The Curriculum Principle

"A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics and well articulated across the grades."

– NCTM (2000, p. 14)

The research of Campbell (1995) and others has shown that *all* children, including those who have been traditionally under-served, can learn mathematics when they have access to high-quality instruction and instructional materials that promote their learning.

Since the passage of Public Law 94-142 in 1975 and its reauthorisation as the Individuals with Disabilities Education Improvement Act (IDEA) in 2004, students in the US with a variety of disabilities are increasingly being taught mathematics in inclusive classrooms. In fact, the majority of students with disabilities are now in regular classrooms for at least a portion of each school day. According to the work of Truelove, Holaway-Johnson, Leslie and Smith (2007), when teachers implement instructional strategies designed to help those with learning disabilities, *all students* – not just those with disabilities – will likely benefit.

STUDENT DISPOSITIONS

During the primary year levels, students often acquire individual views and dispositions toward the learning of mathematics that last for the rest of their lives. Such dispositions as curiosity, cooperation and perseverance are personal habits that play a key role in future success with mathematics both in school and beyond.

An important question to ask is, "Why is it important to take student dispositions into account?" The answer may lie in the work of Dossey, Mullis, Lindquist and Chambers (1988), based on various national assessments. They found that students who enjoy mathematics and perceive its relevance have higher proficiency scores than students with more negative perspectives. They also found that students become less positive about mathematics as they proceed through school; both confidence in and enjoyment of mathematics appear to decline as students progress from primary to secondary school.

One implication of this research is that mathematics instruction should not only enable students to learn skills and understandings but also promote the *desire* to use what has been learned. According to Lannin, Arbaugh, Barker and Townsend (2006), "Part of the process of learning and solving problems includes making errors that, if examined, can lead to further mathematical insight" (p. 182). Lannin and colleagues, and others, believe that teachers should guide students to think and reflect about their errors through a process of recognising, attributing and reconciling.

This book – based on a philosophy of using error analysis with targeted interventions that are meaningful, along with follow-up instructional games and activities – is designed to promote *positive* learning experiences and favourable student dispositions towards mathematics.

> "Students who have developed a productive disposition are confident in their knowledge and ability. They see that mathematics is both reasonable and intelligible and believe that, with appropriate effort and experience, they can learn."
>
> – Kilpatrick, Swafford and Bradford (2001, p. 133)

> "When a child gives an incorrect answer, it is especially important for the teacher to assume that the child was engaged in meaningful activity. Thus, it is possible that the child will reflect on his or her solution attempt and evaluate it."
>
> – Yackel, Cobb, Wood, Wheatley and Merkel (1990, p. 17)

> "If the student is misbehaving out of frustration with an activity, assisting the child with the activity will be more effective than punitive measures in correcting the behavior."
>
> – Truelove, Holaway-Johnson, Leslie and Smith (2007, p. 339)

Finally, children with emotional and behavioural disorders (EBD) often present a variety of challenges to educators. EBD students are especially prone to frustration when performing complex tasks. Guetzloe (2001) and others suggest that *nonaggressive* strategies be used with EBD students to encourage them to stay in class and in school.

ACTIVATING PRIOR KNOWLEDGE

According to Steele (2002) and many others, teachers should review prerequisite skills or concepts no matter how long ago they were taught. Such review is even more important for students who have memory deficits, because they may quickly forget previously mastered skills, or they may have significant gaps in their knowledge.

> "One of the most reliable findings from research is that students learn when they are given opportunities to learn. Providing an opportunity to learn means setting up the conditions for learning that take into account students' entry knowledge, the nature and purpose of the tasks and activities, and so on."
>
> – Hiebert (2003, p. 10)

According to the TIMSS (Trends in International Math and Science Study), teachers in the United States tend to do most of the mental work of introducing, explaining and demonstrating new concepts – and 60% of the time, they do not link new ideas with other concepts and activities. In Japan, where students scored near the top on the TIMSS, *teachers made explicit connections* in 96% of the lessons (U.S. Department of Education, 1996).

The Intervention Activities in this book build on students' prior knowledge by using familiar concepts and tools to develop new content. For example, familiar *place value* concepts are embedded as a key vehicle to develop the algorithms for each operation. Familiar *addition and multiplication tables* are used to reinforce subtraction and division facts, respectively.

REPRESENTATIONS

"The term representation refers both to process and to product" (National Council of Teachers of Mathematics, 2000, p. 67). As a process, it refers to creating in one's mind a mental image of a mathematical idea. As a product, it refers to a physical form of that idea, such as a manipulative, an illustration or even a symbolic expression. Why is the idea of representation so important? Simply stated, *the more ways a student can think about a mathematical concept, the better that student will understand the underlying mathematical idea.*

> **The Representation Standard**
>
> "Instructional programs from prekindergarten through grade 12 should enable all students to
>
> - create and use representations to organize, record, and communicate mathematical ideas;
> - select, apply, and translate among mathematical representations to solve problems;
> - use representations to model and interpret physical, social, and mathematical phenomena."
>
> – NCTM (2000, p. 67)

A Concrete → Semiconcrete → Abstract Model of Instruction: A number of studies suggest that concept development is strong when students begin with a tactile, hands-on model (concrete), move to the use of illustrations of those objects (semiconcrete) and finally move to a symbolic algorithm (abstract). Psychologist Jerome Bruner (1966) referred to those stages as *enactive, iconic* and *symbolic.* Through his research, Bruner theorised that students learn mathematics better when their lessons progress through those three stages. Miller and Hudson (2007) found that such a three-stage model helps students with learning disabilities master concepts involving whole numbers, fractions and algebra. Many of the intervention activities in this book are designed so that students first encounter manipulatives, then refer to drawings of those objects, and finally develop computational proficiency by connecting those representations to an abstract algorithm.

Research: Hands-On Activities; Manipulatives; Diagrams

In a study of over 7000 students, Wenglinsky (2000) found that students whose teachers conduct hands-on learning activities outperform their peers by more than 70% of a year level in maths on the National Assessment of Educational Progress (NAEP).

In a meta-analysis of 60 research studies, Sowell (1989) found that for students of all ages, maths achievement is increased and students' attitudes towards maths are improved with the long-term use of manipulative materials.

Goldin (2003), in analysing many research studies, concluded that "bona fide representational power does not stand in opposition to formal proficiency, but, rather, strengthens it" (p. 283).

Ferrucci, Yeap and Carter (2003) found, from their observations of Singapore schools and curricula, that modelling with diagrams is a powerful tool for children to use to enhance their problem-solving and algebraic reasoning skills.

ESTIMATION AND MENTAL MATHS

Estimation involves a process of obtaining an approximate answer (rather than an exact answer).

Mental maths involves a process of obtaining an exact answer in your head.

"Estimation relates to every important mathematics concept and skill developed in elementary school."

— Reys and Reys (1990, p. 22)

Traditionally, estimation and mental maths have been thought of as supplemental skills. However, based on surveys of adults, Carlton (1980) found that most of the mathematics used in everyday living relies far more on estimation and mental computation than on traditional computation.

Also, traditionally, mental maths and estimation have been taught *after* students master pencil-and-paper computation. However, Kilpatrick and colleagues (2001) found not only that children can learn to compute mentally and to estimate *before* learning formal pencil-and-paper computational procedures but also that mental maths and estimation activities prior to formal work with computation actually enhance the learning of computation.

This book describes and integrates a variety of strategies to use for estimation. Front-end (with adjustment), rounding and compatible numbers are all suggested as ways to check for the *reasonableness* of results. Because some teachers may not be as familiar with front-end estimation as, say, with rounding, this book provides instructional material on using front-end estimation for each operation. Front-end estimation focuses on the "front-end" digit of a number – the digit in the place-value position that contributes the most to the final answer. This method often provides better estimates than the rounding method because numbers that are close to the "middle" of a range (such as 352 or 349) are not dramatically rounded up (400) or down (300). Such an example is illustrated on the right.

Although many struggling students find the *rounding* method to be difficult, most traditional textbooks teach that method as the primary way to form estimates. To make rounding accessible to more students, this book includes a lesson titled "Roller Coaster Rounding." The roller coaster model provides a way for students *visualise* the rounding process. This book also provides instruction for using *compatible* (nice) *numbers* to estimate results. Students should be allowed to use the estimation strategy with which they are most comfortable, and they should be given ample opportunities to discuss those strategies with one another.

To promote fluency with mental maths for addition and subtraction, this book provides Intervention Activities that use an "empty (open) number line" as a model. A growing body of research has reported on an international trend toward its use. According to Bobis (2007), students using the empty number line concluded that it is "easier to learn and remember than the pencil-and-paper method" essentially because the actions performed on an empty number line represent the *student's* thinking (p. 411).

According to O'Loughlin (2007), "Some children need a model like the open number line to keep a record of their counting and help them think while experimenting with patterns and relationships and thus developing number sense" (p. 134). Further, many students have difficulty learning the standard subtraction algorithm. The standard algorithm, shown at the right, is often the *starting point* of subtraction instruction. The following section addresses the benefits of using alternative algorithms with struggling students.

Estimation by Rounding (to the Nearest 100)

$$\begin{array}{r} 253 \\ +\,455 \\ \hline \end{array} \longrightarrow \begin{array}{r} 300 \\ +\,500 \\ \hline 800 \end{array}$$

Front-End Estimation

$$\begin{array}{r} 253 \\ +\,455 \\ \hline \end{array} \longrightarrow \begin{array}{r} 2 \\ +\,4 \\ \hline 6 \end{array}$$

Add the front-end digits. Since 2 hundreds + 4 hundreds = 6 hundreds, the sum is at least 600.

Now adjust: Since 53 + 55 is about 100, an estimate would be 600 + 100, or about 700.

Exact Answer

253 + 455 = 708

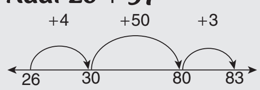
So, the estimate produced by front-end estimation (about 700) is closer to the exact answer than the estimate produced by rounding (800).

"When students have regular opportunities to estimate, share orally, evaluate, compare their approaches, and transfer strategies to new settings, they feel challenged and, ultimately, empowered."

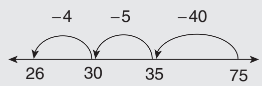
– Rubenstein (2001, p. 443)

Using an Empty Number Line

Add: 26 + 57



Think: 26 + 4 = 30;
30 + 50 = 80; 80 + 3 = 83



Subtract: 75 – 49

Think: 75 – 40 = 35;
35 – 5 = 30; 30 – 4 = 26

Standard Subtraction Algorithm

$$\begin{array}{r} {\scriptstyle 6\ 15} \\ \not{7}\,\not{5} \\ -\ 4\ 9 \\ \hline 2\ 6 \end{array}$$

Research: Asking Children to Compare Estimation Strategies

Star, Kenyon, Joiner and Rittle-Johnson (2010), citing several research studies, concluded "a promising approach that has emerged from research in mathematics education and cognitive psychology emphasizes the role of comparison – comparing and contrasting multiple solution methods – in helping students learn to estimate" (p. 557).

Research: Children's Thinking on Mental Maths

Many students think that mental maths is nothing more than doing a traditional algorithm in your head. Reys and Barger (1994) found that teaching and practising the written algorithms before doing any mental maths actually increases the likelihood that children will think that way.

Research: Using an Empty Number Line for Mental Maths

Beishuizen (2001) found that students are able to successfully use the empty number line for two-digit addition and subtraction. The empty number line aids students in recording and making sense of a variety of solution strategies.

History of the Word *Algorithm*

Around 780–850 C.E., Muhammad ibn-Musa al-Khwarizmi wrote Book on Addition and Subtraction After the Method of the Indians (title translated from the Arabic). In his book, solutions to problems are given in steps, or recipes. The word for these recipes, algorithm, is derived from the Latin that begins with Dixit Algorismi, or "al-Khwarizmi says."

– Pickreign and Rogers (2006, pp. 42–47)

ALTERNATIVE ALGORITHMS

An *algorithm* is "a precise, systematic method for solving a class of problems" (Maurer, 1998, p. 21). In school mathematics, students generally learn a traditional algorithm for each operation that is quite efficient. However, according to Van de Walle (2001), "Each of the traditional algorithms is simply a clever way to record an operation for a single place value with transitions ('trades', 'borrows' or 'carries') to an adjacent position" (p. 171). Although many students experience success using traditional algorithms, some students do not.

Unfortunately, some teachers give struggling students *more* instruction and practice using the same algorithms for which those students have already demonstrated failure. According to Ellis and Yeh (2008), "the traditional algorithms used for subtraction and multiplication are very efficient but not very transparent – they do not allow students to see why the methods work. When students learn traditional algorithms by rote, they often come to think of this as *the* way to do arithmetic rather than as *one* way among many" (p. 368).

These students often continue to struggle with the following kinds of questions:

- When multiplying with renaming, why do you multiply the next digit in the multiplicand before you add, rather than after?
- When multiplying by a 2-digit number, why do you move the second partial product one space to the left?
- In long division, why do you multiply and subtract as part of the process?
- In long division, what is the reason for the use of the phrase "bring down"?

"The depressing thing about arithmetic badly taught is that it destroys a child's intellect and, to some extent, his integrity. Before they are taught arithmetic, children will not give their assent to utter nonsense; afterwards they will. Instead of looking at things and thinking about them, they will make wild guesses in the hopes of pleasing the teacher."

– Sawyer (1943)

This book provides extensive, step-by-step Intervention Activities to address the traditional algorithms. However, the Intervention Activities also include *alternative algorithms* for each operation. According to Lin (2007/2008), alternative methods help students "understand how other algorithms work and prompt them to think more deeply about numbers and equations" (p. 298). It should be noted that alternative algorithms not only are effective with students who struggle with traditional algorithms but are also effective with *all* students up front – and may be used *instead* of those algorithms (or in addition to them). Many textbook programs include alternative algorithms with their materials because they benefit *all* students.

"The standard algorithms used in the United States are not universal. . . . As our schools become more and more diverse, it is important that students' knowledge from their home cultures is valued within the classroom. Having students share alternative methods for doing arithmetic is one way to do so and honors the knowledge of their parents and community elders."

– Ellis and Yeh (2008, p. 368)

One such alternative algorithm is for two-digit multiplication that uses grids to help find partial products. Englert and Sinicrope (1994) noted that "although the time spent in developing the multiplication algorithm using this visual approach is greater than the time needed to use a more traditional approach, less time is needed for review and reteaching. Students are able to attach meaning to the multiplication algorithm" (p. 447).

An important premise of this book is that for each operation, the dual benefits of *teaching for understanding* and *saving time* can be achieved by using meaningful alternative algorithms.

DIFFERENTIATING INSTRUCTION

According to Stiff, Johnson and Johnson (1993), "if all students were the same, a teacher's job would be simple – and boring. Researchers would develop one comprehensive theory of learning; teachers would simply follow the recipe to produce high levels of success for 'all' students. The challenge is to find the combination of strategies that enable all students to reach their full potential" (p. 12).

More Than One Way to Perform an Operation

"Most people have been taught only one way, so they quite naturally assume that there is only one way. The realization that there are many possible procedures to follow when operating on numbers can change the way that people think of mathematics."

– Sgroi (1998, p. 81)