Problem-Solving Strategies for *Efficient* and *Elegant* Solutions

Grades 6–12

A Resource for the Mathematics Teacher

Alfred S. Posamentier
Stephen Krulik

Afterword by Nobel Laureate Herbert A. Hauptman

Second Edition

Hawker Brownlow
EDUCATION

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Problem 1.2

In a room with 10 people, everyone shakes hands with everybody else exactly once. How many handshakes are there?

Solution A

Let's use our *visual representation* strategy, by drawing a diagram. The 10 points (no 3 points of which are collinear) represent the 10 people. Begin with the person represented by point *A*.

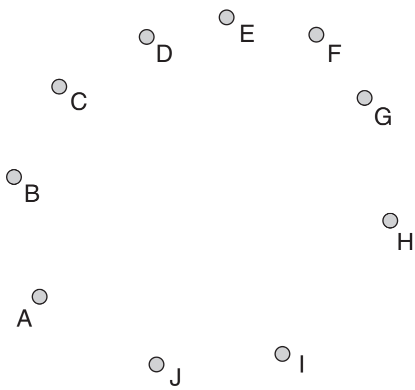

Figure 1.12

We join *A* to each of the other 9 points, indicating the first 9 handshakes that take place.

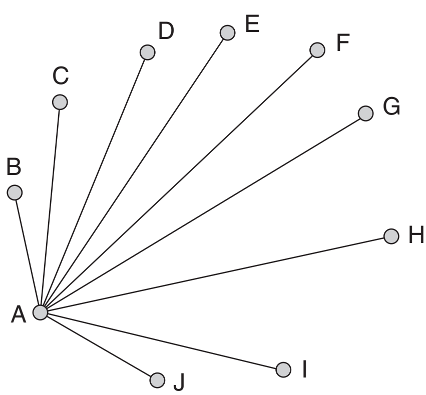

Figure 1.13

Problem 3.7

How many digits are there in the expression $(111,111,111)^2$? What is the middle digit?

Solution

Some students may attempt to "beat the problem to death," that is, actually perform the indicated multiplication. This can be done, but requires extreme care, because the typical student calculator does not accept a nine-digit number.

Let's see if we can solve this problem by *looking for a pattern*:

1 digit	1^2	= 1	= 1 digit; middle digit = 1
2 digits	11^2	= 121	= 3 digits; middle digit = 2
3 digits	111^2	= 12321	= 5 digits; middle digit = 3
4 digits	1111^2	= 1234321	= 7 digits; middle digit = 4
⋮	⋮	⋮	⋮
9 digits	$111,111,111^2$	= 12345678987654321	= 17 digits; middle digit = 9

There are 17 digits in the product; the middle digit is 9.

Although the symmetry of the original problem seemed to imply the presence of a pattern, it still was not an obvious method of approach. Only through practice will students try to find a pattern as a possible method of solution.

Problem 3.8

How many pairs of vertical angles are formed by 10 distinct lines, concurrent through a point?

Solution

Students often attempt to draw a large, accurate figure showing the 10 concurrent lines. They then attempt to actually count the pairs of vertical angles. This is rather confusing, however, and they can easily lose track of the pairs of angles under examination.

Instead, we can make use of our *search for a pattern* strategy. Let's start with a simpler case, gradually expand the number of lines, and see if a pattern emerges.

If we start with one line, we get 0 pairs of vertical angles.

Two lines produce 2 pairs of angles: 1-3 and 2-4 (Figure 3.7a).

Three lines produce 6 pairs of angles: 1-4; 2-5; 3-6; 1, 2-4, 5; 2, 3-5, 6; 1, 6-3, 4 (Figure 3.7b).

Four lines produces 12 pairs of vertical angles: 1-5; 2-6; 3-7; 4-8; 1, 2-5, 6; 2, 3-6, 7; 3, 4-7, 8; 4, 5-8, 1; 1, 2, 3-5, 6, 7; 2, 3, 4-6, 7, 8; 3, 4, 5-7, 8, 1; 4, 5, 6-8, 1, 2 (Figure 3.7c).

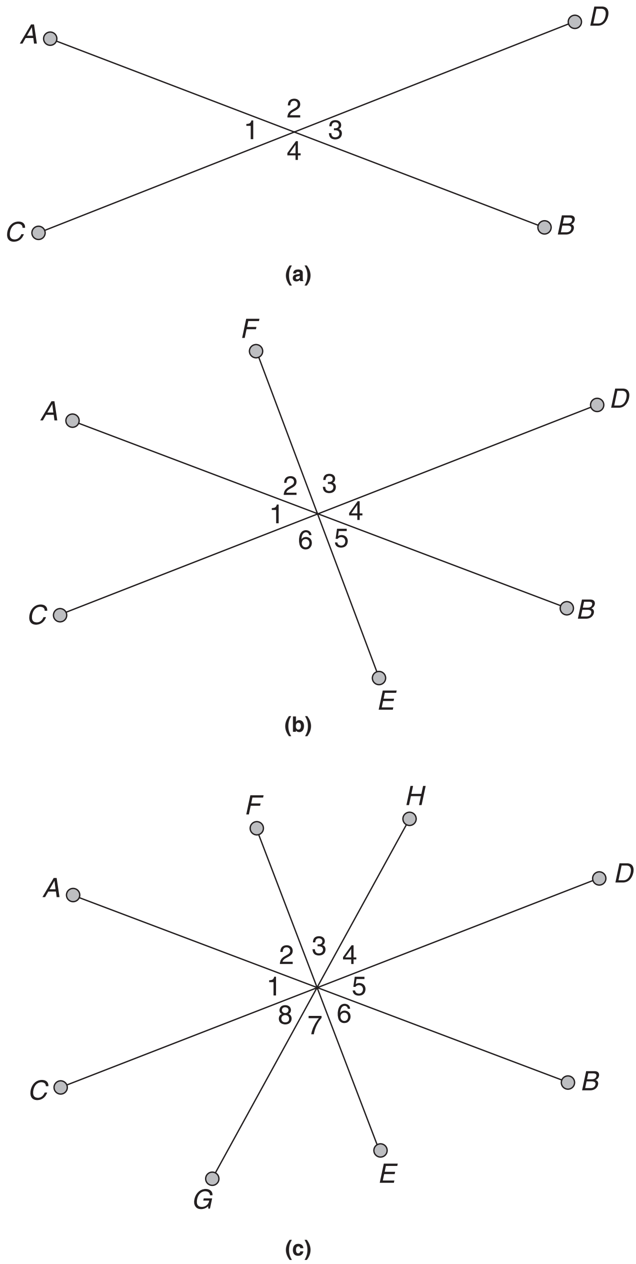

Figure 3.7

Problem 4.25

Find the value of the following (without a calculator, of course!):

$$\frac{10^2 + 11^2 + 12^2 + 13^2 + 14^2}{365}.$$

Solution

Rather than doing the indicated arithmetic, consider *adopting a different point of view*, that is, rewriting each of the terms as follows:

$$\frac{(12-2)^2 + (12-1)^2 + 12^2 + (12+1)^2 + (12+2)^2}{365}.$$

By expanding each of the binomial squares, we get

$$\frac{(12^2 - 48 + 4) + (12^2 - 24 + 1) + 12^2 + (12^2 + 24 + 1) + (12^2 + 48 + 4)}{365}.$$

This can be simplified by combining terms as follows:

$$\frac{5(12^2) + 4 + 1 + 1 + 4}{(5)(73)}.$$

This gives us

$$\frac{720 + 10}{(5)(73)} = 2.$$

Although a calculator computation would have been just as efficient, this is perhaps more elegant!

Problem 4.26

Find the measure of the angle *ABC* formed by the two diagonals of the given cube.