

Problem-Solving  
Strategies for  
*Efficient and  
Elegant Solutions*  
Grades 6–12

**A Resource for the  
Mathematics Teacher**

Alfred S. Posamentier  
Stephen Krulik

Afterword by Nobel Laureate Herbert A. Hauptman

**Second Edition**



---

# Contents

Preface	vii
About the Authors	ix
<b>1. Introduction to Problem-Solving Strategies</b>	<b>1</b>
<b>2. Working Backwards</b>	<b>17</b>
The <i>Working Backwards</i> Strategy in Everyday Life Problem-Solving Situations	18
Applying the <i>Working Backwards</i> Strategy to Solve Mathematics Problems	19
Problems Using the <i>Working Backwards</i> Strategy	21
<b>3. Finding a Pattern</b>	<b>39</b>
The <i>Finding a Pattern</i> Strategy in Everyday Life Problem-Solving Situations	40
Applying the <i>Finding a Pattern</i> Strategy to Solve Mathematics Problems	42
Problems Using the <i>Finding a Pattern</i> Strategy	43
<b>4. Adopting a Different Point of View</b>	<b>69</b>
The <i>Adopting a Different Point of View</i> Strategy in Everyday Life Problem-Solving Situations	70
Applying the <i>Adopting a Different Point of</i> <i>View</i> Strategy to Solve Mathematics Problems	71
Problems Using the <i>Adopting a Different</i> <i>Point of View</i> Strategy	72
<b>5. Solving a Simpler Analogous Problem</b>	<b>97</b>
The <i>Solving a Simpler Analogous Problem</i> Strategy in Everyday Life Problem-Solving Situations	97
Applying the <i>Solving a Simpler Analogous Problem</i> Strategy to Solve Mathematics Problems	98

Problems Using the <i>Solving a Simpler Analogous Problem</i> Strategy	100
<b>6. Considering Extreme Cases</b>	<b>117</b>
The <i>Considering Extreme Cases</i> Strategy in Everyday Life Problem-Solving Situations	118
Applying the <i>Considering Extreme Cases</i> Strategy to Solve Mathematics Problems	118
Problems Using the <i>Considering Extreme Cases</i> Strategy	121
<b>7. Making a Drawing (Visual Representation)</b>	<b>141</b>
The <i>Making a Drawing</i> (Visual Representation) Strategy in Everyday Life Problem-Solving Situations	141
Applying the <i>Making a Drawing</i> (Visual Representation) Strategy to Solve Mathematics Problems	142
Problems Using the <i>Making a Drawing</i> (Visual Representation) Strategy	143
<b>8. Intelligent Guessing and Testing (Including Approximation)</b>	<b>167</b>
The <i>Intelligent Guessing and Testing</i> (Including Approximation) Strategy in Everyday Life Problem-Solving Situations	168
Applying the <i>Intelligent Guessing and Testing</i> (Including Approximation) Strategy to Solve Mathematics Problems	168
Problems Using the <i>Intelligent Guessing and Testing</i> (Including Approximation) Strategy	170
<b>9. Accounting for All Possibilities</b>	<b>189</b>
The <i>Accounting for All Possibilities</i> Strategy in Everyday Life Problem-Solving Situations	189
Applying the <i>Accounting for All Possibilities</i> Strategy to Solve Mathematics Problems	190
Problems Using the <i>Accounting for All Possibilities</i> Strategy	191
<b>10. Organizing Data</b>	<b>203</b>
The <i>Organizing Data</i> Strategy in Everyday Life Problem-Solving Situations	203
Applying the <i>Organizing Data</i> Strategy to Solve Mathematics Problems	204
Problems Using the <i>Organizing Data</i> Strategy	206
<b>11. Logical Reasoning</b>	<b>225</b>
The <i>Logical Reasoning</i> Strategy in Everyday	

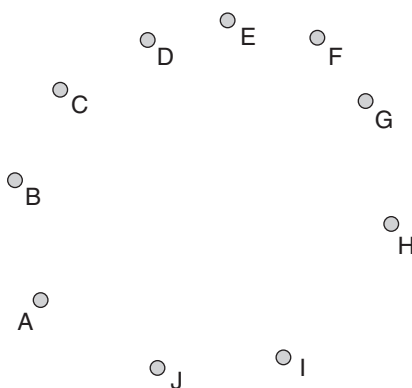
Life Problem-Solving Situations	225
Applying the <i>Logical Reasoning</i> Strategy to Solve Mathematics Problems	226
Problems Using the <i>Logical Reasoning</i> Strategy	226
<b>Afterword</b>	<b>243</b>
<i>Herbert A. Hauptman</i>	
<b>Sources for Problems</b>	<b>249</b>
<b>Readings on Problem Solving</b>	<b>257</b>

**Problem 1.2**

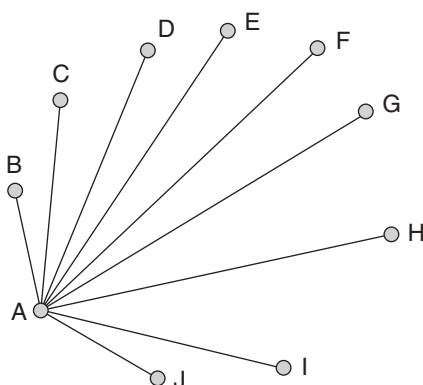
In a room with 10 people, everyone shakes hands with everybody else exactly once. How many handshakes are there?

**Solution A**

Let's use our *visual representation* strategy, by drawing a diagram. The 10 points (no 3 points of which are collinear) represent the 10 people. Begin with the person represented by point A.

**Figure 1.12**

We join A to each of the other 9 points, indicating the first 9 handshakes that take place.

**Figure 1.13**

**Problem 3.7**

How many digits are there in the expression  $(111,111,111)^2$ ? What is the middle digit?

**Solution**

Some students may attempt to “beat the problem to death,” that is, actually perform the indicated multiplication. This can be done, but requires extreme care, because the typical student calculator does not accept a nine-digit number.

Let’s see if we can solve this problem by *looking for a pattern*:

1 digit	$1^2$	= 1	= 1 digit; middle digit = 1
2 digits	$11^2$	= 121	= 3 digits; middle digit = 2
3 digits	$111^2$	= 12321	= 5 digits; middle digit = 3
4 digits	$1111^2$	= 1234321	= 7 digits; middle digit = 4
⋮	⋮	⋮	⋮
9 digits	$111,111,111^2$	= 12345678987654321	= 17 digits; middle digit = 9

There are 17 digits in the product; the middle digit is 9.

Although the symmetry of the original problem seemed to imply the presence of a pattern, it still was not an obvious method of approach. Only through practice will students try to find a pattern as a possible method of solution.

**Problem 3.8**

How many pairs of vertical angles are formed by 10 distinct lines, concurrent through a point?

**Solution**

Students often attempt to draw a large, accurate figure showing the 10 concurrent lines. They then attempt to actually count the pairs of vertical angles. This is rather confusing, however, and they can easily lose track of the pairs of angles under examination.

Instead, we can make use of our *search for a pattern* strategy. Let’s start with a simpler case, gradually expand the number of lines, and see if a pattern emerges.

If we start with one line, we get 0 pairs of vertical angles.

Two lines produce 2 pairs of angles: 1-3 and 2-4 (Figure 3.7a).

Three lines produce 6 pairs of angles: 1-4; 2-5; 3-6; 1, 2-4, 5; 2, 3-5, 6; 1, 6-3, 4 (Figure 3.7b).

Four lines produces 12 pairs of vertical angles: 1-5; 2-6; 3-7; 4-8; 1, 2-5, 6; 2, 3-6, 7; 3, 4-7, 8; 4, 5-8, 1; 1, 2, 3-5, 6, 7; 2, 3, 4-6, 7, 8; 3, 4, 5-7, 8, 1; 4, 5, 6-8, 1, 2 (Figure 3.7c).

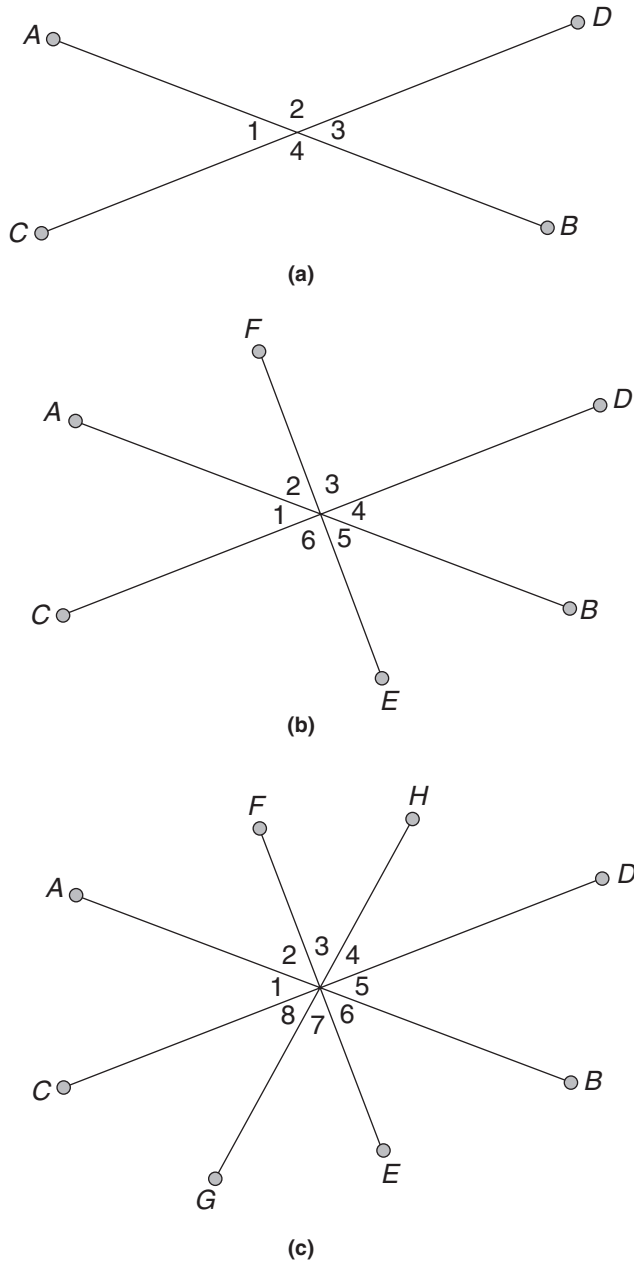


Figure 3.7

**Problem 4.25**

Find the value of the following (without a calculator, of course!):

$$\frac{10^2 + 11^2 + 12^2 + 13^2 + 14^2}{365}.$$

**Solution**

Rather than doing the indicated arithmetic, consider *adopting a different point of view*, that is, rewriting each of the terms as follows:

$$\frac{(12 - 2)^2 + (12 - 1)^2 + 12^2 + (12 + 1)^2 + (12 + 2)^2}{365}.$$

By expanding each of the binomial squares, we get

$$\frac{(12^2 - 48 + 4) + (12^2 - 24 + 1) + 12^2 + (12^2 + 24 + 1) + (12^2 + 48 + 4)}{365}.$$

This can be simplified by combining terms as follows:

$$\frac{5(12^2) + 4 + 1 + 1 + 4}{(5)(73)}.$$

This gives us

$$\frac{720 + 10}{(5)(73)} = 2.$$

Although a calculator computation would have been just as efficient, this is perhaps more elegant!

**Problem 4.26**

Find the measure of the angle  $ABC$  formed by the two diagonals of the given cube.