YEARS
5–9

MATHS EXPLORATIONS

Ratios, Proportions & Similarity

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Teacher's Guide

GOALS

The explorations in this series were developed through years of work with talented middle years maths students. They are designed to

- engage students in the excitement of mathematical discovery
- deepen students' understanding of middle years maths concepts
- help students become flexible, creative, disciplined mathematical thinkers
- improve mathematical communication skills
- explore connections between maths concepts
- develop patience, perseverance and stamina in solving math problems
- provide depth and challenge for a variety of needs and interests
- enable students to work collaboratively and independently
- offer opportunities for further exploration.

THE EXPLORATIONS

This book contains problems that will challenge virtually any middle years maths student. The explorations are self-differentiating. As students progress through an activity, the level of challenge or depth increases. A few students may finish. Most will reach a stopping point.

Students will progress according to their age, mathematical experience, persistence, capacity and the amount of time available. Some may want to give up quickly. A few may insist on completing every problem even if they do not understand them well. A simple rule of thumb is that students should spend most of their time working on problems that are just beyond their comfort level. When they reach these problems, they should stick with them for a long time. They will learn more from thinking deeply about one or two problems than from rushing to finish a lot of them.

The problems in each exploration are grouped into three stages. Each successive stage extends the depth or the level of challenge. The end of a stage is a convenient place to pause and consider whether to continue. To help you decide, every stage starts with a brief description of the problems it contains along with information about the knowledge students will need and a summary of what they will learn.

Every exploration has a number of features to support your work with students: the Problems, some Conversation Starters, the Solutions and Algebra Connections. Some of them have an additional feature called a Classroom Vignette. These features are described next.

THE PROBLEM PAGE

Each Problem page has an "opener" and a list of directions. The opener is a sort of teaser that sets up the problem situation without telling students what to do. The directions fill in the details.

You may use the Problem page as a handout. My favourite approach is to cover the directions before I copy it so that students see only the opener on the handout. This is much more fun than giving them all of the information upfront. As we discuss the opener, students actively participate in creating the task by predicting (and suggesting) what the directions will be. This helps them learn that maths is about asking questions, not just giving answers. It also allows me to identify possible points of confusion at the start.

At the end of this discussion, we finalise the directions. Based on students' ideas and their learning goals, we either modify the original directions or use them as is.

Some of the Problem pages have Testing the Waters or Diving Deeper questions at the bottom of the page. Testing the Waters questions are less complex versions of the main problem. They make it accessible to more students. If students are not making progress on the original problem, you can suggest that they begin with Testing the Waters. Even if they get no further, they will learn important new ideas.

The Diving Deeper questions are just what they sound like – an opportunity to explore in more depth. Many of these are more challenging than the original problem. Others point students to related questions or topics of interest.

THE CONVERSATION STARTERS PAGE

The Conversation Starters are observations and questions that can or should arise in discussion. Sometimes, your students will come up with these. At other times, you will need to work them into the conversation. Their purpose is to help you guide your students' thinking without telling them how to do the problems.

You will probably not use all of the Conversation Starters on the page. Choose those that best fit your students' needs and learning goals, or follow up on the ones that your students initiate. You do not have to use them in any particular order, but I have tried to organise them in a way that is likely to follow the flow of discussion. The Conversation Starters near the end of the page are often extensions of the main ideas.

I have written the Conversation Starters as "I wonder" questions and "I notice" statements. In keeping with the philosophy of encouraging independent thinking,

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many of the "I wonder" questions are not answered. I have left them open for you and your students to think about. Even when you are not sure of the answer, the question may point your thinking in a useful direction. In some cases, especially with the items near the end of the page, I raise a question out of curiosity, and I may not know the answer myself.

By the way, "I wonder" questions may pop up in the Solutions, too! You are never done with a good problem. There are always more questions to ask!

THE SOLUTIONS PAGE

In writing the Solutions, I have tried to strike a balance between giving enough detail to support your work and not giving so much that it makes the problems look harder than they are. Most solutions are one or two pages long. There are two main reasons for their length: (1) I include many student strategies, and (2) the problems contain a lot of ideas.

Please keep in mind that longer answers do not necessarily mean more classroom time. In some cases, I have simply shown many ways to think about the problem. On the other hand, a one- or two-line answer in the Solutions may represent a lot of thought and discussion. Although a solution may look short on paper, there is no short cut for the effort and thinking that goes into finding it.

The Solutions are not the final word! You and your students may discover more efficient or more interesting strategies than I have shown. You will have insights that have not occurred to me. Each time I teach the explorations, I learn something new about the maths.

THE ALGEBRA CONNECTIONS PAGE

I have written most of these explorations assuming that students have a certain level of knowledge of algebra – that they can understand, interpret and even create algebraic expressions and equations, but they have learned few rules for manipulating them. Because your students will vary in their knowledge and experience, I have included an Algebra Connections page at the end of most activities. It has three purposes:

- To help you see connections to students' future learning.
- To give students a chance to try their hand at algebraic processes and reasoning before they are taught all of the "steps" in Year 7 maths!
- To offer students who have studied algebra a chance to apply their skills to the problem.

If the Algebra Connections page does not seem relevant to your purposes, you may ignore it. You will not need it for other explorations. But I hope you will glance at the connections between the content you are teaching and the concepts your

students will study when you are no longer their teacher. You may gain valuable perspectives that inform your teaching. And if you feel comfortable doing so, allow your students to play with some of the algebraic expressions and equations from time to time (without teaching them the rules)! This is a powerful way to integrate their understanding of numbers and variables.

THE CLASSROOM VIGNETTES

Some explorations have a sample conversation called a Classroom Vignette. Most of these are taken from actual conversations that I have had or witnessed with students. In some cases, I have combined ideas from conversations that took place over a couple of days. I have streamlined the flow of the discussions to make them more readable. And, of course, I have made up the names of the teachers and students. To make the conversations easier to write and read, there are usually just three or four students talking.

The main purpose of the vignettes is to give you a feel for how conversations might look when you are helping students to think independently and to develop conceptual understanding. I have chosen to write them on topics and problems that I felt could use a closer look, often because the Conversation Starters and the Solutions did not quite seem to capture certain key ideas.

I have not tried to make the vignettes illustrate every strategy for conducting effective discussion. I have worked harder to make them realistic than to make them perfect. My main hope is that these examples flesh out some important mathematical ideas for you and that they give you a starting point for thinking about ways to engage your students in conversations that help them make sense of math in a deep way.

EIGHT MOTIVATION STRATEGIES

1. **Let students know what to expect.** Tell students that the problem or activity will take time. Let them know that they will sometimes get stuck and that their work will probably not be perfect. Give them a time frame, and let them know how you will support them.
2. **Redefine success.** Tell students that success is not just about speed and accuracy. Let them know that you value effort, progress, creativity, insight and clear communication – in short, you care more about learning than perfection.
3. **Praise effort over ability.** Praising effort over ability encourages risk-taking. Seeing intelligence as a quantity that changes through effort empowers students to reach their potential. Carol Dweck develops these ideas in her book, *Mindset* (2007).

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4. **Focus on process more than answers.** Respond to right and wrong answers in a similar manner, focusing on the mathematical ideas and the opportunity to learn something new. Show students that you value an interesting question as much as an accurate answer.
5. **Offer emotional support.** Some talented maths students do not accept real challenges due to a fear of not looking "smart". They may not be accustomed to feeling frustration. They need help managing these feelings, especially if maths has always come easily to them.
6. **Offer meaningful responses to written work.** You do not have to write a lot, just a few specific and thoughtful comments on students' completed work to let them know that you have read and considered their ideas.
7. **Allow students to collaborate.** In addition to the enjoyment of social interaction, collaboration makes students feel safer taking risks. And, of course, they have more success, because they are sharing ideas!
8. **Debrief.** After you finish a problem (or set of problems), talk about it before you move on. Kids love this! Share answers and strategies. Talk about what went right and what went wrong. Summarise key ideas. Discuss things that are still confusing. Think of new questions to ask.

TEACHING STRATEGIES

Maths is about ideas! Of course, skills are necessary, too, but without a conceptual foundation, students will not be able to apply skills to problems or use them to support further learning.

Shifting from a focus on procedural skills to a more balanced approach that recognises the key role of ideas requires thinking in new ways. The strategies in the right hand column of Table 2 show how to use these explorations to support conceptual understanding and to infuse new depth and meaning into your students' learning.

CLASSROOM DISCUSSIONS

The explorations are designed so that students may spend much of their time working without direct instruction. But they will need to talk about the problems with you and with each other. You may need to be creative to find time for discussion, especially if you have a classroom with a wide range of needs, but it is worth the effort. The more that you and your students talk about the maths, the more progress they will make and the more they will learn.

Equally important is what happens during conversation. Fortunately, you do not have to explain how to do the problems. That is your students' job! Yours is first to ensure that they have the basic knowledge needed to approach the problem and then to orchestrate conversation so that they learn from each other. The

TABLE 1

Teaching Strategies

Traditional Strategies	Strategies That Support Deep Learning
Prepare students for guided practice by clearly explaining procedures using worked examples.	Expect students to learn by thinking their way through challenging problems that engage them with the concepts.
Teach skills first. Then have students apply them to story problems.	Use problem-solving as a means of teaching concepts and skills.
Mark homework by marking answers right or wrong.	Respond to students' work by writing comments related to their thinking.
Study answers in advance so that you can explain them clearly to the students.	Be ready to discuss unexpected strategies and learn new ideas from students.
Know the process you want students to use.	Assign tasks that can be solved in many ways. Discuss advantages and disadvantages of different methods.
Have every student do the same questions.	Differentiate goals and assignments based on students' learning needs.
Have fixed deadlines for assignments.	Be flexible with due dates if students run into unexpected difficulties or want to explore further.

Conversation Starters give examples of questions and observations that move a discussion forward without telling students what to do. When in doubt, ask rather than answer, and say less rather than more.

To make conversations productive, classrooms must have a culture of curiosity and respect. All contributions to discussion are valuable, because all have the potential to create learning. Give students plenty of "wait time" before and after you call on them so that they have time to think and to formulate their responses. Ask them to speak in a strong voice and to direct their comments to the class rather than to you. Have them question, repeat or rephrase each other's statements as needed. Have them agree or disagree – always explaining why. To facilitate, you may record and organise their ideas on the board. Rephrase their statements yourself for clarification if necessary, but always check that you have understood their ideas correctly. To learn more about these and other techniques for questioning and orchestrating classroom discussions, see Chapin, O'Connor and Anderson (2013) and Smith and Stein (2011).

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Exploration 1

The Incredible Shrinking Universe

As space enthusiasts know, the universe is not really shrinking; it is expanding! But, in this exploration, students discover that they can use maths to bring astronomical measurements down to a human scale and to imagine the incomprehensible!

The large numbers in these problems are the most interesting but also the most challenging feature for my students. I generally allow them to use calculators. At the same time, I encourage them to use their knowledge of place value to simplify the calculations, and I insist that they estimate before they calculate – to think about what sort of an answer they expect before they "crunch the numbers".

I placed this exploration at the beginning of the book because my students are able to enjoy and understand it before they have studied proportional relationships in depth. Talented younger students often succeed using their intuition and their knowledge of map scales. The activity provides them with an excellent set of experiences to support their emerging understanding of deeper concepts surrounding proportionality. If your students struggle early with the activity, return to it when they have had a little more experience with ratios or large numbers.

STAGE 1

Many students have studied our solar system and are curious to learn more, but the sizes and distances involved are hard to imagine! Stage 1 will help them understand how the sizes of the planets compare.

Some students may not feel comfortable choosing their own scale and developing their own strategies. They may ask what the "best" scale and the "best" strategy are. It takes time and patience for students to learn that there is not always a best answer – and that even when there is, they are capable of discovering it for themselves! These are exactly the skills they need to learn in order to apply maths to real-world situations.

What Students Should Know

- Understand multiplication and division of decimals.
  - Multiply and divide fractions and decimals using efficient written strategies and digital technologies (ACMNA154)
- Convert between different metric units of measurement.
  - Convert between common metric units of length, mass and capacity (ACMMG136)
- Have experience working with large numbers.
  - Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written strategies and appropriate digital technologies (ACMNA100)
- Read and interpret a scale on a map (recommended).
  - Use simple scales, legends and directions to interpret information contained in basic maps (ACMMG090)

What Students Will Learn

- Choose an appropriate scale for a scale model.
- Use multiplication and division to create a scale model.
- Apply knowledge of place value to calculate with large numbers.
- Use intuition about scale models to further develop proportional reasoning skills.

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