YEARS
5–9

MATHS EXPLORATIONS

Probability & Statistics

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Teacher's Guide

GOALS

The explorations in this series were developed through years of work with talented middle years maths students. They are designed to

- engage students in the excitement of mathematical discovery
- deepen students' understanding of middle years maths concepts
- help students become flexible, creative, disciplined mathematical thinkers
- improve mathematical communication skills
- explore connections between maths concepts
- develop patience, perseverance and stamina in solving maths problems
- provide depth and challenge for a variety of needs and interests
- enable students to work collaboratively and independently
- offer opportunities for further exploration.

THE EXPLORATIONS

This book contains problems that will challenge virtually any middle years maths student. The explorations are self-differentiating. As students progress through an activity, the level of challenge or depth increases. A few students may finish. Most will reach a stopping point.

Students will progress according to their age, mathematical experience, persistence, capacity and the amount of time available. Some may want to give up quickly. A few may insist on completing every problem even if they do not understand them well. A simple rule of thumb is that students should spend most of their time working on problems that are just beyond their comfort level. When they reach these problems, they should stick with them for a long time. They will learn more from thinking deeply about one or two problems than from rushing to finish a lot of them.

The problems in each exploration are grouped into three stages. Each successive stage extends the depth or the level of challenge. The end of a stage is a convenient place to pause and consider whether to continue. To help you decide, every stage starts with a brief description of the problems it contains along with information about the knowledge students will need and a summary of what they will learn.

Every exploration has a number of features to support your work with students: the Problems, some Conversation Starters, the Solutions and Algebra Connections. These features are described next.

THE PROBLEM PAGE

Each Problem page has an "opener" and a list of directions. The opener is a sort of teaser that sets up the problem situation without telling students what to do. The directions fill in the details.

You may use the Problem page as a handout. My favourite approach is to cover the directions before I copy it so that students see only the opener on the handout. This is much more fun than giving them all of the information upfront. As we discuss the opener, students actively participate in creating the task by predicting (and suggesting) what the directions will be. This helps them learn that maths is about asking questions, not just giving answers. It also allows me to identify possible points of confusion at the start.

At the end of this discussion, we finalise the directions. Based on students' ideas and their learning goals, we either modify the original directions or use them as is.

Some of the Problem pages have Testing the Waters or Diving Deeper questions at the bottom of the page. Testing the Waters questions are less complex versions of the main problem. They make it accessible to more students. If students are not making progress on the original problem, you can suggest that they begin with Testing the Waters. Even if they get no further, they will learn important new ideas.

The Diving Deeper questions are just what they sound like – an opportunity to explore in more depth. Many of these are more challenging than the original problem. Others point students to related questions or topics of interest.

THE CONVERSATION STARTERS PAGE

The Conversation Starters are observations and questions that can or should arise in discussion. Sometimes, your students will come up with these. At other times, you will need to work them into the conversation. Their purpose is to help you guide your students' thinking without telling them how to do the problems.

You will probably not use all of the Conversation Starters on the page. Choose those that best fit your students' needs and learning goals, or follow up on the ones that your students initiate. You do not have to use them in any particular order, but I have tried to organise them in a way that is likely to follow the flow of discussion. The Conversation Starters near the end of the page are often extensions of the main ideas.

I have written the Conversation Starters as "I wonder" questions and "I notice" statements. In keeping with the philosophy of encouraging independent thinking, many of the "I wonder" questions are not answered. I have left them open for you and your students to think about. Even when you are not sure of the answer, the question may point your thinking in a useful direction. In some cases, especially with the items near the end of the page, I raise a question out of curiosity and I may not know the answer myself.

By the way, "I wonder" questions may pop up in the Solutions, too! You are never done with a good problem. There are always more questions to ask!

THE SOLUTIONS PAGE

In writing the Solutions, I have tried to strike a balance between giving enough detail to support your work and not giving so much that it makes the problems look harder than they are. Most solutions are one or two pages long. There are two main reasons for their length: (1) I include many student strategies and (2) the problems contain a lot of ideas.

Please keep in mind that longer answers do not necessarily mean more classroom time. In some cases, I have simply shown many ways to think about the problem. On the other hand, a one- or two-line answer in the Solutions may represent a lot of thought and discussion. Although a solution may look short on paper, there is no short cut for the effort and thinking that goes into finding it.

The Solutions are not the final word! You and your students may discover more efficient or more interesting strategies than I have shown. You will have insights that have not occurred to me. Each time I teach the explorations, I learn something new about the maths.

THE ALGEBRA CONNECTIONS PAGE

I have written most of these explorations assuming that students have a pre-algebra level of knowledge – that they can understand, interpret and even create algebraic expressions and equations, but they have learned few rules for manipulating them. Because your students will vary in their knowledge and experience, I have included an Algebra Connections page at the end of most activities. It has three purposes:

» To help you see connections to students' future learning.
» To give pre-algebra students a chance to try their hand at algebraic processes and reasoning before they are taught all of the "steps" in algebra class!
» To offer students who have studied algebra a chance to apply their skills to the problem.

If the Algebra Connections page does not seem relevant to your purposes, you may ignore it. You will not need it for other explorations. But I hope you will glance at the connections between the content you are teaching and the concepts your students will study when you are no longer their teacher. You may gain valuable perspectives that inform your teaching. And if you feel comfortable doing so, allow your pre-algebra students to play with some of the algebraic expressions and equations from time to time (without teaching them the rules)! This is a powerful way to integrate their understanding of numbers and variables.

EIGHT MOTIVATION STRATEGIES

1. **Let students know what to expect.** Tell students that the problem or activity will take time. Let them know that they will sometimes get stuck and that their work will probably not be perfect. Give them a time frame, and let them know how you will support them.
2. **Redefine success.** Tell students that success is not just about speed and accuracy. Let them know that you value effort, progress, creativity, insight and clear communication – in short, you care more about learning than perfection.
3. **Praise effort over ability.** Praising effort over ability encourages risk-taking. Seeing intelligence as a quantity that changes through effort empowers students to reach their potential (Dweck, 2007).
4. **Focus on process more than answers.** Respond to right and wrong answers in a similar manner, focusing on the mathematical ideas and the opportunity to learn something new. Show students that you value an interesting question as much as an accurate answer.
5. **Offer emotional support.** Some talented maths students do not accept real challenges due to a fear of not looking "smart". They may not be accustomed to feeling frustration. They need help managing these feelings, especially if maths has always come easily to them.
6. **Offer meaningful responses to written work.** You do not have to write a lot, just a few specific and thoughtful comments on students' completed work to let them know that you have read and considered their ideas.
7. **Allow students to collaborate.** In addition to the enjoyment of social interaction, collaboration makes students feel safer taking risks. And, of course, they have more success, because they are sharing ideas!
8. **Debrief.** After you finish a problem (or set of problems), talk about it before you move on. Kids love this! Share answers and strategies. Talk about what went right and what went wrong. Discuss things that are still confusing. Think of new questions to ask.

TEACHING STRATEGIES

Maths is about ideas! Of course, skills are necessary, too, but without a conceptual foundation, students will not be able to apply skills to problems or use them to support further learning.

Shifting from a focus on procedural skills to a more balanced approach that recognises the key role of ideas requires thinking in new ways. The strategies in the right hand column of Table 2 show how to use these explorations to support conceptual understanding and to infuse new depth and meaning into your students' learning.

CLASSROOM DISCUSSIONS

The explorations are designed so that students may spend much of their time working without direct instruction. But they will need to talk about the problems with you and with each other. You may need to be creative to find time for discussion, especially if you have a classroom with a wide range of needs, but it is worth the effort. The more that you and your students talk about the maths, the more progress they will make and the more they will learn.

Equally important is what happens during conversation. Fortunately, you do not have to explain how to do the problems. That is your students' job! Yours is first to ensure that they have the basic knowledge needed to approach the problem and then to orchestrate conversation so that they learn from each other. The Conversation Starters give examples of questions and observations that move a discussion forward without telling students what to do. When in doubt, ask rather than answer, and say less rather than more.

To make conversations productive, classrooms must have a culture of curiosity and respect. All contributions to discussion are valuable, because all have the potential to create learning. Give students plenty of "wait time" before and after you call on them so that they have time to think and to formulate their responses. Ask them to speak in a strong voice and to direct their comments to the class rather than to you. Have them question, repeat or rephrase each other's statements as needed. Have them agree or disagree – always explaining why. To facilitate, you may record and organise their ideas on the board. Rephrase their statements yourself for clarification if necessary, but always check that you have understood their ideas correctly. To learn more about these and other techniques for questioning and orchestrating classroom discussions, see Chapin, O'Connor and Anderson (2013) and Smith and Stein (2011).

TABLE 2

Teaching Strategies

Traditional Strategies	Strategies That Support Deep Learning
Prepare students for guided practice by clearly explaining procedures using worked examples.	Expect students to learn by thinking their way through challenging problems that engage them with the concepts.
Teach skills first. Then have students apply them to story problems.	Use problem solving as a means of teaching concepts and skills.
Mark homework by marking answers right or wrong.	Respond to students' work by writing comments related to their thinking.
Study answers in advance so that you can explain them clearly to the students.	Be ready to discuss unexpected strategies and learn new ideas from students.
Know the process you want students to use.	Assign tasks that can be solved in many ways. Discuss advantages and disadvantages of different methods.
Have every student do the same questions.	Differentiate goals and assignments based on students' learning needs.
Have fixed deadlines for assignments.	Be flexible with due dates if students run into unexpected difficulties or want to explore further.

ASSESSING STUDENT LEARNING

The tool on p. 13 is designed to assess concept-focused tasks. It was informed and inspired by many sources: the Common Core State Standards for Mathematical Practice (NGA & CCSSO, 2010), the Process Standards of the National Council of Teachers of Mathematics (NCTM, 2000), the five Proficiency Strands in *Adding It Up* (Kilpatrick, 2001) and a rubric in *Extending the Challenge in Mathematics* (Sheffield, 2003).

You may design your own scoring system. I use a 5-point scale in each category.

5 evidence of learning beyond the level of course standards
4 evidence of learning at the level of course standards
3 evidence of learning approaching the level of course standards
2 evidence of learning below the level of course standards
1 evidence of learning significantly below the level of course standards
0 little or no evidence of progress toward meeting course standards

In my classes, students who are new to the explorations often receive 2s and 3s at first. As the school year progresses, they receive mainly 3s and 4s with an occasional 5. Students and parents appreciate the opportunity to identify specific areas of strength and goals for improvement. But no numerical scoring system will ever replace the value of a few thoughtful written comments related to students' ideas!

Of course, you may also incorporate criteria such as legibility, organisation, mechanics (spelling, punctuation and grammar), etc. But above all, your scoring system should reflect the central goal of mathematical learning.

THE AUSTRALIAN CURRICULUM

The Maths Explorations series provides teachers with the perfect resource to extend students' learning with the Australian Curriculum: Mathematics. Each exploration in this book, and the series, is tailored to engage students in the mathematics content, as well as promote the use of practical skills. Each of the Australian Curriculum: Mathematics key ideas – understanding, fluency, problem-solving, and reasoning – are not only present in these explorations, but form an integral part of the learning, ensuring that students are actively engaged in applying the skills introduced and emphasised by the curriculum.

While these activities have not been designed to correlate directly with Australian Curriculum: Mathematics content descriptions, the knowledge and skills that students develop as they learn curriculum content form the basis of their understanding in approaching the explorations here. Th e prior knowledge, or "What Students Should Know", information included at the start of each exploration will inform you of the Australian Curriculum: Mathematics content that will need to be established before students attempt the activity.

The activities in this book are the ideal extension for gifted students using the Australian Curriculum: Mathematics, and will ensure that advanced learners remain engaged with their work in the classroom while practising essential skills.

MATHEMATICAL COMMUNICATION: PART 1

Learning to communicate mathematically offers two key benefits for students. It helps them to develop their own thinking and to communicate with others. This page focuses on the first reason. When I ask young primary students why they believe it is important to explain their thinking, they usually mention both reasons. When I ask older students, they tend to focus on the second reason. I suspect that the more they learn to think of maths only as numbers and calculations, the less value they place on thinking and writing.

You have probably experienced the challenge of trying to convince some of your students to write their ideas down. They may take pride in their ability to do it all in their heads at lightning speed. This type of intuition is wonderful, but it is not always reliable, and it is not enough. When students are working on problems that are sufficiently demanding to be worth their time, there is usually too much information to manage mentally. They must write as they work in order to remember what they have done, to clarify their thoughts, to visualise relationships, to recognize and extend patterns, and to identify and correct errors. I recommend that from the beginning students have paper and pencil in front of them at all times when they are solving challenging problems. In my classes, we call it "thinking paper" rather than scrap paper in order to emphasise its purpose and its importance.

A colleague of mine who is a high school English teacher tells her students that "fuzzy writing means fuzzy thinking". I cannot think of a truer statement for maths class. The effort you take to write clearly helps you to think more clearly!

MATHEMATICAL COMMUNICATION: PART 2

In order to express their ideas when solving deep and challenging problems, your students may need to expand their idea of what it means to "show work". Many important ideas cannot be captured in mathematical symbols alone. There are three common ways to communicate mathematical ideas: words, symbols (numbers, equations, etc.) and diagrams.

If students struggle with putting words on paper, suggest that they speak their ideas aloud and transfer them to paper. Of course, they will need to make some changes, but at least this gets them started. I often have students who insist that they do not know what to write – but are able to speak their thoughts perfectly clearly!

A few tips for using words:

- Use the word "it" sparingly, and always explain what "it" is!
- Can family or friends understand what you have written? If not, then rewrite.
- Be concise. More is not always better.

Students may be familiar with showing their thinking using numbers and equations, but there are still a couple of pitfalls. Labels or explanations of what numbers mean are important. Also, avoid "run-on" maths sentences such as $13 - 7 = 6 \times 2 = 12$. This statement is false, and it shows a misunderstanding of the "=" symbol. It should be written $(13 - 7) \times 2 = 12$ or as two separate equations: $13 - 7 = 6$ and $6 \times 2 = 12$.

Diagrams can sometimes communicate ideas more clearly than words or symbols. They may even replace whole sentences or paragraphs! The key is to include all of the important information without cluttering them with unnecessary or distracting detail.

EXAMPLES OF USING THE EXPLORATIONS

Example 1: Ms Kava teaches gifted maths pull-out groups of 5 to 10 students. Each group meets once per week during its regularly scheduled maths time.

- She coordinates with the classroom maths teachers to select activities that align to course content.
- Students work on explorations during the pull-out. They alternate between partner work and whole-group discussion.