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Introduction

Most people realize they need to know arithmetic, but far fewer understand why they should know algebra. The need to know algebra is not as obvious as the need to be proficient in arithmetic. Yet, algebra is the language of the generalization of arithmetic. Should you need to do something once, then arithmetic usually suffices; however, if you need to do something repetitively, then you need algebra as a tool for generalizing the steps you do arithmetically, regardless of the numbers involved.

- Algebra is the language through which patterns are described.
- Algebra is the language used to describe relationships between quantities.
- Algebra is the language that is used to solve certain types of arithmetic problems.
- Algebra helps people develop new ways of thinking.

Robert Moses, founder of the Algebra Project, and others contend that algebra is a gatekeeper course and that the right to take algebra is a civil rights issue. Too often underserved students—including those living in poverty, those with special needs, and many immigrant, emergent bilingual, and black students—are excluded from the grade eight algebra course, yet when given an experiential approach to algebra, these students are able to demonstrate proficiency. The Algebra Project engaged students in doing algebra. Grade eight students went for rides on the Massachusetts Bay Transit Authority in Boston to model positive and negative integers and were able to demonstrate a greater understanding of the relevance of algebra after all their experiences. We agree with this approach to teaching algebra but recognize its constraints, so we wrote this book, which is filled with activities, modeling, and problem solving that can be done in classrooms.

We hope that this book will be used by teachers of prealgebra or algebraic concepts and/or a traditional algebra course, as well as when introducing functions. We include resources, materials, problems, and games designed to be engaging for students and aligned with the Expressions and Equations, Functions, and High School Algebra domains.

How to Use This Book

Each of the thirty lessons in this book identifies the focal domain and standard(s) for each lesson. Each lesson includes common misconceptions and challenges that many students face together with suggestions for how a teacher might prevent those misconceptions from developing. The main lesson itself provides examples of how different teachers interact with students in their classrooms as they guide their students toward meeting their learning goals. These classroom scenarios provide suggestions to teachers for how they might assign a particular problem or activity, how to include formative assessment strategies, and how to group students. This section also provides fertile material for discussion within a mathematics department meeting or in professional learning communities. Discussing how and why student misconceptions arise is important as teachers strive to avoid partial understandings and erroneous ideas about concepts.

The "In the Classroom" sections are either representations of what happened in one specific classroom or a combination of effective teaching seen in multiple classrooms. The quotes from students are real. We have met and listened to extraordinary students explaining how they are thinking about a problem or what problem-solving strategies they use. We have observed students struggle to pose a question that might help them understand more deeply the mathematics they are investigating, and we have witnessed high energy in many classrooms while students are working through the activities. Some modules include tasks or problems that might be used in activity centers, for individualized challenges or remediation, or for homework. The center activities are designed to allow students to work at their own level, to challenge students in need of a challenge, or to reinforce previously learned concepts. The "In the Classroom" section also suggests instructional strategies, activities, and samples of student thinking, including connections to the eight Standards for Mathematical Practice (SMPs).

The section on "Meeting Individual Needs" suggests some strategies for students who are struggling or who need a greater challenge. The suggestions in this section include some remediation strategies or further activities to challenge those students who need more in-depth work and some very rigorous problems for students who like to go deeper into a concept than what is typically possible in a heterogeneous classroom.

The final section of each lesson is the "Additional Readings/Resources." We understand how busy teachers are and that finding good research articles or classroom vignettes can be very time-consuming, so we included some references for each lesson. We also suggest that all teachers visit the National Council of Teachers of Mathematics website, which has a plethora of wonderful resources, interactive activities, and readings.

We hope you agree that by embracing the old Chinese proverb, "Tell me, and I will forget. Show me, and I will remember. Involve me, and I will understand," all our students will have access and opportunity to enjoy a rich and fulfilling experience while studying algebra.

We encourage you to keep this resource on your desk or next to your plan book so that you will have these ideas at your fingertips throughout the year. Of course, no single book can contain all of the mathematics pertaining to any given topic. For more detailed discussions about mathematical and teaching issues raised in this book (and elsewhere), visit our blog *Accessible Algebra* at stenhouse.com/accessiblealgebra.

Additional Reading/Resources

Collins, Anne, and Linda Dacey. 2010. *Zeroing in on Number and Operations: Key Ideas and Common Misconceptions.* Portland, ME: Stenhouse.

———.2011. *The Xs and Whys of Algebra: Key Ideas and Common Misconceptions.* Portland, ME: Stenhouse.

———.2013. *It's All Relative: Key Ideas and Common Misconceptions.* Portland, ME: Stenhouse.

Expressions

1. Order of Operations

DOMAIN: **Expressions and Equations**

STANDARDS: **6.EE.2a.** Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

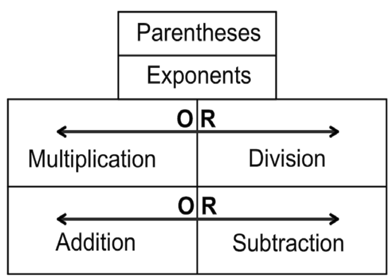
6.EE.3. Apply the properties of operations to generate equivalent expressions.

7.EE.1. Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.

Potential Challenges and Misconceptions

Many students struggle with simplifying expressions with several operations because they do not understand that they are actually writing equivalent expressions as they conduct each operation. Too often students are asked to memorize the mnemonic device PEMDAS without understanding it or what its limitations are. Many find that using PEMDAS causes them to incorrectly simplify expressions, because the mnemonic neglects to include *multiply OR divide from LEFT to RIGHT* and *add OR subtract from LEFT to RIGHT*. It is most helpful to provide students with a template that helps them organize their work systematically.

In the Classroom



To help students visualize the order in which expressions are simplified, one teacher uses a “hopscotch” diagram. He draws three large classroom-sized diagrams on the floor with liquid shoe polish and labels the operations as shown. He also provides his students with heavyweight plain paper and tells them to make their own diagram to refer to as they write equivalent expressions.

Once all students have the hopscotch diagram at their tables, he projects an expression on the board. He invites three student volunteers to use the floor diagram, moving to the appropriate steps and writing the equivalent expressions. (Notice, the term *simplify the expression* is not being used. This is intentional since every step of the “simplification process” is writing equivalent expressions.) Each student volunteer records the steps and resulting mathematical expression after each operation is conducted. While the three students are working on the floor diagrams, the rest of the class is going through the same process on their desk-sized representations.

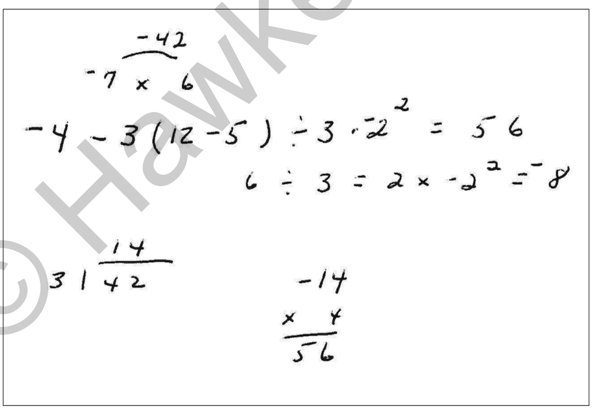
After an allotted period of time, this teacher asks the three volunteers for the final expression and records it on the board. He then asks the class if anyone got a different answer or if they did it differently. He quickly asks a volunteer to describe the order in which they completed the steps used. He repeats this activity until every student has an opportunity to use the floor-sized diagrams, and as the activity continues, he makes the expressions more and more complicated. See *Classroom Expressions* on page A1-2 in the appendix.

Next, this teacher groups his students into triads and assigns the number 1, 2, or 3 to each student. He invites the number 1s to the board to record an expression that he dictates. He instructs these students to complete the first computation and write an equivalent expression. These students then sit down, and the number 2s go to the board to do the next step. This continues until the equivalent expression cannot be simplified further. While students rotate turns at the board, they are also completing the expression at their seats so that every student completes every expression. This teacher repeats this process, dictating expressions and inviting students to work on writing equivalent expressions until he's confident that students understand the process. Next, he assigns the reproducible *In What Order?* on page A1-3 of the appendix.

Students in grade seven were asked to write equivalent expressions for the expression $^-4 - 3(12 - 5) \div 3 \cdot {^-2^2}$ (item 4 from *In What Order?*). Classroom teachers predicted the greatest difficulty would be computing $^-2^2$, which did give many students difficulty but, surprisingly, not as many students as expected. Here are samples of student work showing some of the challenges and misconceptions.

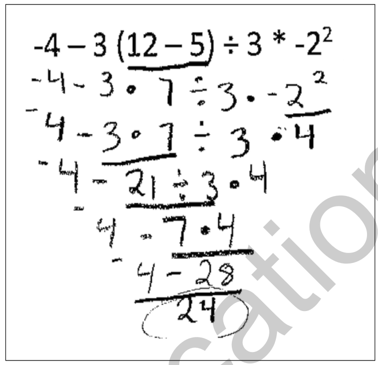
Student A

Notice that Student A attempts to do each calculation above the original expression. This lacks the structure and organization that is helpful to most students: writing each equivalent expression underneath the previous one. This student inaccurately subtracts 12 – 5 to get 6 instead of 7 but, more importantly, subtracts before multiplying the quantity of 12 – 5 by 3. This student also adds an equal sign to indicate the solution, perhaps thinking that he or she is solving an equation rather than writing a series of equivalent expressions or "simplifying an expression."



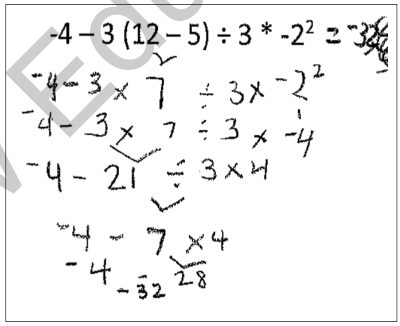
This teacher gives students the choice of using lined paper or the *In What Order? Graphic Organizer* on page A1-4 in the appendix. He cautions students who choose lined paper to record each expression on a separate line and directly below the one before. This provides the teacher a more logical and sequential way of understanding his students' thinking.

Student B



This student correctly follows the order of operations but makes errors calculating with negative numbers. When simplifying $^{-}2^2$, he or she makes the classic error of raising $^{-}2$ to the second power instead of raising 2 to the second power then multiplying by $^{-}1$. This student also computes $^{-}4 – 28$ and gets a +24.

Student C



This student computes using the correct order of operations and accurately simplifies $^{-}2^2$. Unfortunately, when the student moves on to the fourth step, he or she neglects the negative sign: instead of multiplying $^{-}7$ by $^{-}4$, the student multiplies $^{-}7$ by +4. This student also inappropriately adds an equal sign to the expression.

As a result of seeing the students' work, this teacher began using the *In What Order? Graphic Organizer* to help his students work systematically when using the order of operations.

Periodically, this teacher assigns the order of operations activity, *Equivalent Expressions*, in various activity centers to allow his students to practice and hone their mathematical skills. *Equivalent Expressions* can be found on page A1-6 in the appendix. Note that teachers should cut out the cards and group them as described on the reproducible before giving them to students. Students select the equivalent expressions strips and glue them in the appropriate order in their notebooks.

Meeting Individual Needs

Some students benefit from using a graphic organizer that lists the operations in the order they should be calculated, line by line. This not only helps them choose the appropriate operation but also helps them organize and structure the results of forming equivalent expressions. (See *In What Order? Graphic Organizer* in the appendix.) Assign the reproducible *In What Order?*, together with the graphic organizer, to help students internalize the hierarchy represented by order of operations.

Additional Reading/Resources

Blackwell, Sarah B. 2003. "Operation Central: An Original Play Teaching Mathematical Order of Operations." *Teaching Mathematics in the Middle School* 9 (1): 5.

Equations

11. Activity Centers

DOMAIN: **Expressions and Equations**

STANDARD: **6.EE.C.** Represent and analyze quantitative relationships between dependent and independent variables.

Potential Challenges and Misconceptions

All students need to recognize that there are linear and nonlinear relationships and that they can be represented in tables, graphs, expressions, and equations. In addition, it is important for students to distinguish between continuous and discrete data as represented in tables and graphs. Many students struggle with these concepts because most or all their elementary experience has revolved around bar graphs, and if they did draw a linear graph, they always connected the points. To connect points on a graph or not is a new consideration for most students. All students benefit from varied experiences in contextual situations as they meet the challenge of determining whether points are connected or not. For example, it is not probable to have half a marble, but it is probable to have half a cup of milk. Students also need multiple experiences determining how the data in a table look in a graph and vice versa. The inclusion of activity centers in the algebra classroom is an effective way to challenge, remediate, or meet the needs of grade-level students.

In the Classroom

One teacher projects *What Kind of Graph Am I?* from page A2-2 in the appendix, and distributes the reproducible to his students. He instructs his students to examine each graph and determine whether the graph is linear, quadratic, exponential, absolute value, or something else. He encourages his students to collaborate on the identification process and instructs them to record the reason for their selections. These activities engage students in several of the Standards for Mathematical Practice.

Prior to class, the teacher has set up a variety of activity centers. The centers present activities that review a previous model for representing expressions or represent a different relationship and result in a different graphical representation. At each center, students work in small groups to record the data from their activities on easel-sized paper and make their own conclusions about the shape of the graphs. This teacher allows time for his students to complete at least two activities per class period, with all four completed in two days. After all students have completed tables and graphs on easel-sized paper for each station, this teacher facilitates a gallery walk. He tells his students to match the graphs to the corresponding activity.

Activity Center One

This center has five vases of various shapes and sizes, rulers, a pitcher with colored water (or rice), a small scoop, and paper towels. Students are required to fill each vase with scoops of colored water and record the number of scoops and the height of the water in the vase after each scoop of water is added. The resultant graphs are typically nonlinear, unless the vase is a cylinder, with the graphs taking on the shape of the vase.

Activity Center Two

This center has a set of cards on the table. Students are directed to match the graph with its corresponding description See *What Kind of Graph Am I?* from page A2-2 in the appendix.

Activity Center Three

This center will focus on building factor lattices using variables. The center has a box of either gumdrops or dots candy (small Styrofoam pieces or play dough also work) and toothpicks. Students make a factor lattice for xy^3. Directions are on *Factor Lattices 2*, page A2-3 in the appendix. Students use the factor lattice to determine the greatest common factor and least common multiple between various algebraic expressions. Students will also represent the expressions using Venn diagrams as an alternative representation.

Activity Center Four

This center focuses on modeling exponential growth situations. The center has a stack of paper and scissors on the table. The directions for this center instruct students to take one sheet of paper, cut it in half, and record in a table the number of cuts and number of resulting pieces of paper. Next they stack the pieces and cut the stack in half. They repeat this cutting process until they cannot cut the paper again. Students are directed to make a table to show the relationship between the number of cuts and the number of pieces of paper that result. They graph the data in the table and write an algebraic expression that best models the relationship.

Activity Center Five

This center has a container of Algeblocks with instructions to model a variety of expanded and factored representations. Students are told to build a model for each expression, sketch the model, and write an algebraic expression that best represents the results of the operations performed in the model. Sample expressions can be found in *Modeling Algebraic Expression*s on page A1-22 in the appendix.

Meeting Individual Needs

The inclusion of the activity centers allows this teacher to provide differentiated instruction to students who need more support and simultaneously provide a challenge for students who need a deeper exploration of algebraic concepts. Teachers should assign students to the appropriate centers to ensure they are being challenged, reviewing difficult concepts, or reinforcing learning that may be fragile.