Uncomplicating ALGEBRA *to Meet Common Core Standards in Math, K–8*

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CONTENTS

PREFACE

ORGANIZATION OF THE BOOK

This resource is intended to help teachers improve student success in learning algebra by sharing approaches that will lead to a deeper and richer understanding of the subject.

The resource is organized by grade level around the Common Core State Standards for Mathematics (CCSSM) that are related to algebraic thinking. The grades covered in this resource begin with Kindergarten, where the first relevant standard is found in the Operations and Algebraic Thinking domain, and end with Grade 8, where the focus is on working with linear equations and functions. For each section, a portion of the relevant standard is presented, followed by a delineation of important underlying ideas associated with that portion of the standard, as well as some Good Questions to Ask to bring those underlying ideas out.

The discussions of underlying ideas include

- background on the mathematics of the standard,
- suggestions for appropriate representations of the specific mathematical ideas,
- suggestions for explaining the ideas to students, and
- cautions about misconceptions or situations to avoid.

Following each set of underlying ideas is a group of Good Questions to Ask that can be used for classroom instruction, student practice, or assessment. Among the questions are many open questions, as well as more directed conceptual questions that might be supplemental to what teachers normally are provided in the resources they use. The Common Core State Standards for Mathematical Practice underlie the content throughout and are explicitly mentioned in a number of instances.

For Whom Is This Book Useful and Why?

This resource is designed to aid math teachers of Kindergarten–Grade 5 in building a solid foundation for student work in algebra in the middle grades and to aid teachers of Grades 6–8 in preparing students for work in algebra in the secondary grades. It is also intended to serve as a resource for math coaches in assisting classroom teachers in their transition to teaching mathematics within the more demanding framework of the Common Core State Standards. I expect this book to be helpful as well to preservice teachers as they prepare themselves to understand and teach math in a way that will foster a deep level of understanding in their students.

Considering the Bigger Picture

While I would hope that all users would read the entire book, I particularly encourage this approach for math coaches and preservice teachers. For grade-level or grade-band teachers, I suggest reading the Introduction and the grade-level sections that most directly apply for their particular groups of students, but also becoming acquainted with the mathematics related to algebra taught in grades directly below and above their groups. Because students in any classroom possess different levels of knowledge, in order to differentiate instruction appropriately, teachers must be aware of missing prerequisite knowledge, as well as suitable directions for moving forward.

Lastly, I hope that using this book helps make algebra make more sense both to the readers and to their students.

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Introduction

INCREASED FOCUS ON ALGEBRA

Students' success or lack of success in early algebra can have a significant effect on their futures (Usiskin, 1995). Algebra is often required for graduation from high school. It is also seen as a critical course for opening doors for many future careers. In fact, one of the tasks of the National Mathematics Advisory Panel to the president of the United States was to identify the skills needed for students to learn algebra (National Mathematics Advisory Panel, 2008). It is widely accepted that to achieve the current U.S. goal of algebra for all, students in elementary and middle schools must have better preparatory experiences than has historically been the case (Cai & Knuth, 2005).

WHAT IS ALGEBRA?

Although many view algebra as math that you do with letters, the topic of algebra is much more complex than that. There is value in looking at how different researchers define algebra to make sense of how algebra manifests itself in the Common Core State Standards for Mathematics.

For example, Usiskin (1988) described four different notions of what algebra is:

- A way to generalize and formalize arithmetic: for example, using the algebraic equation $ab = ba$ to indicate that any two numbers can be multiplied in either order; or $a(-b) = -ab$ as a means to indicate that the product of any number and the opposite of another one is the opposite of the product of the two numbers; or $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ as a way to indicate the rule for multiplying fractions, again no matter what the numerators and denominators are.
- A procedure for solving certain kinds of problems (e.g., problems like this one: If one amount is 50 more than twice another, and the two amounts total 300, what is each amount?).
- The study of relationships among quantities or variables, for example, $a = 2p$ as a means of describing the number of arms, a, of p people; or $P = 2(l + w)$ as a means of describing the perimeter of a rectangle as double the sum of its length and width. Also involved in working with these

relationships is insight into what they imply. For example, knowing that $a = 2p$ tells us that as p increases, so does a. Knowing that $P = 2(l + w)$ tells us that the sum of the length and width of a rectangle must be half its perimeter.

- The study of structures with certain inherent rules, for example, we factor $x^2 - 9$ by using the rules of symbol manipulation.

Some of these approaches to algebra manifest themselves in the CCSSM standards for grades K–8, particularly the generalizing of arithmetic concepts and the study of relationships among quantities and variables. Usiskin's (1988) other notions of algebra tend to be more significant in the secondary grades.

Generalization is a significant focus even in the early grades. It involves a deliberate extension from particular situations and often involves justification. For example, students generalize when they realize that it is not just that 2 + 3 = 3 + 2 and 5 + 8 = 8 + 5, but that any two numbers can be added in any order, and understand why. Or a student might notice that both 4 × 9 and 2 × 18 are ways to express 36, but then generalize to the concept that when we multiply two numbers to achieve a particular positive whole number product, if one factor increases, the other decreases. Ellis (2007) points out that generalization is complex, often involving reasoning and communication, and that the ability to generalize grows with more and more opportunities to generalize, which often occurs when students work with patterns.

Variables are related in the earlier grades not only when students create formulas using some measurements of a shape to determine other measurements but also when they consider how various groups of numbers relate, for example, how the multiples of 5 relate to the multiples of 10.

The National Council of Teachers of Mathematics (NCTM, 2000) lists four somewhat different organizing themes for algebra: (1) understanding patterns, relations, and functions; (2) representing and analyzing mathematical situations and structures; (3) using mathematical models to represent and understand quantitative relationships; and (4) analyzing change in various contexts. These themes relate to and overlap Usiskin's (1988) notions and manifest themselves clearly in the CCSSM. Work on pattern in Grades 3–5 leads to generalization, a hallmark of algebra. Using equations to describe both numerical and measurement situations even as early as Grade 1 eventually leads to an examination of how variables are related. Consideration of mathematical models and quantitative relationships occurs at almost all grade levels, with significant attention to the meaning of equations. Analyzing change becomes more prominent in the middle grades, where students explore changes in variables, often using tables of values and graphs.

Because the various aspects of algebra touch on so many areas, standards that require algebraic thinking are found in many strands of the CCSSM, including Number and Measurement.

GENERAL REASONS STUDENTS STRUGGLE WITH ALGEBRA

Historically, the separation of arithmetic and algebra in instructional resources and in teacher instruction for Grades K–8 might have unintentionally interfered with student success in algebra. Students did not really look at algebra as a way to generalize the concepts they dealt with in arithmetic, yet that is an important aspect of algebra, as discussed above. It is telling that the CCSSM, which have been formalized fairly recently, use the subdomain of Operations and Algebraic Thinking within the Number strand in Grades K–5, in recognition of the value of helping teachers and students see the interconnection between number and algebra.

Kieran (2004) suggests that there are critical features that must be included in the integration of arithmetic and algebra to lead to student success in algebra. These include

- A focus on looking at relationships between values and not just on calculating answers,
- A focus on the inverse relationships between addition and subtraction, and multiplication and division, to support equation solving,
- A focus on representing problems and not just on solving them
- A focus on the use of variables along with numbers from an early grade, and
- More attention to the meaning of the equal sign as a description of a relationship or equivalence than as an instruction for getting an answer.

Students who do not develop these focuses will likely struggle more in algebraic situations than those who do. These concepts are all addressed in the CCSSM to build the likelihood of developing success. They are also specifically addressed in this resource in a number of the suggestions and questions provided.

There are many other issues, too, that interfere with success when students are coming to grips with algebraic situations. Some of these issues are rooted in the nature of algebra, whereas others result from missing prerequisite knowledge.

Algebra is abstract from the point of view that it is about generalizations and not specifics. Knowing that $3 \times 4 = 12$, and so does 6×2, is specific. Realizing that, when any two numbers are multiplied, the first can be doubled and the second halved without changing the product is a generalization. Many teachers focus on specifics, and many students do not get past this stage. The CCSSM suggest that teachers encourage generalization.

Students who approach problems in an unsystematic way will have more difficulty than students who are systematic in arriving at a generalization. Students who are scattered in their thinking simply do not recognize the patterns from which they might generalize. Teachers need to help students see the value of organization in detecting relationships.

Algebra requires more abstract thinking than does much work with numbers. To efficiently figure out how to graph, for example, $y = 3x - 2$ requires an understanding of the role of the coefficient of x and the constant in a linear equation. To go from tables of values to appropriate equations requires an ability to observe patterns, make sense of them, and then generalize. This type of abstract thinking requires careful development on the part of a teacher; it is not automatic for many students.

Another important prerequisite to success in algebra is a thorough understanding of addition, subtraction, multiplication, and division. To use an equation to model a problem such as "If I have 20 times as many stamps as Rachel, and I have 420 stamps, how many does Rachel have?" the student requires a deep understanding of what multiplication (or division) means, when it applies, and how to translate between natural language and algebra. In this case, a student without that knowledge might easily just multiply 20×420, seeing both of those numbers and the phrase "times as many" in the question, rather than realizing that the equation is actually $20r = 420$, which makes the question essentially a division problem. Teachers must ensure that students meet and model problems involving all sorts of meanings of operations and experience many opportunities to translate between natural language and algebra.

Algebraic reasoning often requires deduction, that is, considering how knowing one piece of information leads to another. Students without practice in this habit of mind struggle in algebra. For example, students have to understand why, if they know that $x + y = 20$, they also implicitly know that $2x + 2y = 40$, why x must be an integer if y is, and why, if y is a negative integer, then x must be a positive integer. Teachers can facilitate this habit of mind by regularly asking questions that require students to deduce.

Even relatively early work in algebra also requires some reasonable level of comfort with proportional reasoning. Thinking of $3x$ as 3 of the unit x, or of $3x + 2$ as just about the same as $3x$ for large values of x, are examples of thinking proportionally. This ability is fundamental to making sense of even simple algebraic expressions. The literature indicates that many students lack even basic proportional reasoning (Dole, 2010). Development of proportional reasoning is aided by careful teacher attention to it while teaching number, algebra, and measurement.

Yet another reason for difficulties in algebra is students' lack of understanding of the meaning of an equal sign, a critical part of algebraic thinking and a point very specifically addressed in the CCSSM. Many students think of the equal sign as a signal to perform some calculation, rather than seeing it as a way to describe two equivalent expressions or a balance. So, those students, when confronted with the equation $400 \div 2 = \square \times 5$, will assume that $\square = 200$, the answer to $400 \div 2$, rather than 40, the value that would make the two sides of the equation equal. As well, those students are totally confused by an equation like $3x + 2 = 2x + 6$, since they are looking for an answer (i.e., a number) on the right-hand side (Carpenter, Franke, & Levi, 2003; Knuth, Stephens, McNeil, & Alibali, 2006).

Students might also be confused when variables are used in different ways. For example, when a student sees $3 + k = 8$, she or he is usually expected to determine the single unknown value that makes the equation true. However, this is not the case when the student sees any of the following:

- The expression $3 + x$, which describes an infinite set of numbers;
- The function $f(x) = x + 3$, which also describes an infinite set of inputs/outputs; or
- The equation $3 + x = (5 - x) + (2x - 2)$, which is true for any value of x, not just one, since this is a statement of equivalence.

Teachers need to point out these different uses of a variable.

Hallagan (2006) points out that variables make some students so uncomfortable they often do not know how to handle them when they are included in an answer to a question, for example, a question such as "Describe an algebraic expression which means three more than a number." They believe answers should be numbers. Perhaps this is why Booth (1998) indicated that students are less comfortable with algebraic expressions than with equations; with an equation, there is at least something to do. This phenomenon means that a teacher needs to spend extra time on expressions, making the meaning of expressions clear to students. Many suggestions for focusing on expressions are provided in this resource.

Additional obstacles to success in algebra are related to missing or faulty prerequisite knowledge in students. Often this missing knowledge is a solid number sense and/or comfort with operations involving particular types of numbers. For example, solving the equation $\frac{3}{4}x = \frac{5}{8}$ requires competence with multiplication and/or division of fractions. Adding $3n$ to $(-4n)$ requires competence with addition of integers. Solving $3x - 2 = 4(x + 3)$ requires competence with order of operations. Recognizing the difference between $2 - 3x$ and $3x - 2$, or between $4(2x - 3)$ and $8x - 3$, requires an understanding of properties of numbers. Teachers need to be realistic when selecting the algebraic situations they use with students in terms of the prerequisite knowledge possessed by the students.

At the middle school level, another problem for students in understanding algebra could be lack of comfort with graphing, an important aspect of an algebra program once students begin to explore relationships between two variables. Teachers must provide students with experience in analyzing graphs and not just in creating them.