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Introduction to Assessments That Teach

THE TEACHING OF mathematics and assessing the mathematical performance of our students and the effectiveness of our mathematics instructional programs has become a major concern of the mathematics education community and the community as a whole.

This book is a guide to a model of middle years mathematics assessment that helps teachers to aid their students in making the transition to later mathematics, algebra in particular.

The approach to assessment that we take in this book organises this body of mathematical content and these mathematical processes in a way that teachers can find compatible with their thinking and that of their students. In order to introduce this organisation of mathematics, we turn to the question "What is mathematics about?"

WHAT IS MATHEMATICS ABOUT?
THE STRUCTURE OF THE SUBJECT AS SEEN BY BALANCED ASSESSMENT

As in many disciplines, it is possible to identify both content and process dimensions in the subject of mathematics. As in many subjects, the process dimension of mathematical knowledge and understanding refers to general reasoning, problem-formulating and problem-solving skills. These are necessary but not sufficient. In mathematics, much of the process dimension refers to many skills that are not general, but rather mathematics-specific. Because of this breadth of processes – some general and some specific to mathematics – many people tend to lump content and process together when speaking about mathematics, calling it all mathematics *content*.

However, it is important to maintain a distinction here. In part we say this because we believe that this distinction reflects something very deep about the way humans approach mental activity of all sorts. All the languages that people speak have grammatical structures that allow for the parsing of utterances into *noun* phrases and *verb* phrases. It seems that people like to think in terms of *objects* and the *actions* that are carried out on them or by them.

In mathematics we deal with mathematical objects such as fractions, triangles and linear functions, and with the actions that we carry out on these objects. Some of these actions are quite specific mathematical actions, such as multiplication or rotation, and some of them are much more general actions, like modelling or drawing inferences.

The Balanced Assessment model of task development relies on this *object–action* distinction. We have found that the word *object* better captures for teachers and students what ordinarily is meant by such terms as fraction, decimal, triangle, quadratic, variable, function and so on. We have found that the word *action* better captures general cognitive actions such as modelling, manipulating, inferring and so on, as well as such specific mathematical actions such as subtraction, division, exponentiation and rotation.

What are the mathematical objects we wish to deal with? What are the mathematical actions that we carry out on and with these objects? We will try to answer these questions in a way that makes clear the continuity of the subject from the earliest year levels through post-secondary mathematics. Since there are really very few discrete categories of mathematical objects and actions, this approach offers a simple and clear way to view and organise the teaching and learning of mathematics. This is a way of looking at *any* F–12 (and beyond) mathematics curriculum and does not imply either the introduction of new mathematical content or the omission of mathematical content that is traditionally present. It is a way of helping you, the teacher, to think about the subject and make its coherence across the year levels more transparent.

THE OBJECTS OF MATHEMATICS

The first category of mathematical objects we consider is that of number and quantity. Indeed, primary mathematics is largely about these objects, and the actions or operations we carry out with and on them. Some of the maths objects included in number and quantity are:

- integers (positive and negative whole numbers and zero)
- rationals (fractions, decimals and all the integers)
- measures (length, area, volume, time, weight)
- real numbers (π and some other irrationals such as *e*, as well as all the rationals)
- complex numbers (these usually make their first appearance in secondary school)
- vectors and matrices (also typically a secondary school topic)

Along with number and quantity, we introduce very early a concern for another kind of mathematical object, namely, *shape and space*. Objects of this kind, along with the operations that are carried out on and with them, typically appear in the geometry strand of most curricula. Maths objects that are investigated in this domain (again, some of these concepts are encountered in the early years, some much later) include

- well-defined geometric spaces (such as lines/segments, polygons, circles, etc)
- spatial concepts (such as area, volume, connectedness and enclosure)

From the beginning we try to make students aware of *patterns* in the worlds of number and shape. In the primary year levels, patterns and sequences are closely aligned with arrangements, but by the time we reach the middle years, patterns as a mathematical object mature into *function*s, which are rules for generating *output* mathematical objects from one or more *input* objects. Functions, and the actions we carry out on or with them, are the essential content of the topics we call algebra and calculus. Here are some examples of what we mean by pattern and function objects:

- functions on real numbers (linear, quadratic and power functions in the middle years and early secondary school; rational, periodic and transcendental functions later on)
- functions on shapes (rotation, translation, reflection, scaling and dilation, partition, etc)

There are several other kinds of mathematical objects in the mathematics we expect our students to study. These include *chance and data* and *arrangement*. *Chance and data* are concerned with maths objects such as

- relative frequency and probability
- discrete data (e.g. people, cars, dresses, etc) and continuous data (e.g. time, weight, distance, etc)

Although some aspects of data collection, organisation and presentation may be introduced in the early years using tables and charts, rigorous data analysis and notions of probability realistically are not addressable until the middle years and beyond.

In the early year levels, arrangement tends to blend with the study of patterns of numbers and shapes. By the middle years, the topic becomes more discrete; some of the maths objects to be considered are

- permutations and combinations
- graphs
- networks, trees, counting schemes

THE ACTIONS OF MATHEMATICS

As discussed earlier, the actions of mathematics include mathematics-specific actions as well as general problem-solving skills that are needed throughout all aspects of learning and living. We divide these general skills into four categories.

- Modelling / Formulating
- Transforming / Manipulating
- Inferring / Drawing Conclusions
- Communicating

Although these are quite general skills, each of these actions has aspects that are specific to mathematics, as well as aspects that are general in nature. Here are some of these general and specific aspects.

Modelling / Formulating

general problem-solving actions

- observation and evidence gathering
- necessary and / but not sufficient conditions
- analogy and contrast
- deciding, with awareness, what is important and what can be ignored

mathematics-specific actions

- deciding, with awareness, what can / should be mathematised and then doing so
- formally expressing dependencies, relationships and constraints

Transforming / Manipulating

general problem-solving actions

- understanding 'the rules of the game'
- understanding the nature of equivalence and identity

mathematics-specific actions

- arithmetic computation
- symbolic manipulation in algebra and calculus
- formal proofs in geometry

Inferring / Drawing Conclusions

general problem-solving actions

- shifting point of view
- testing conjectures

mathematics-specific actions

- investigation of symmetry and invariance
- investigation of limiting cases (e.g. what number does the sequence .9, .99, .999, .9999, . . . approach?)
- investigation of 'between-ness' – utilising order properties (e.g. asking students to make up a subtraction problem whose answer lies between the answers to the following two subtraction problems: 72 – 28 and 101 – 87, or to find a quadratic function that is everywhere larger than $-x^2 + 2$ and smaller than $2x^2 + 3$; asking, How many such problems can you make? How many such functions can you find? How do you know?)

Communicating

general problem-solving actions

- making a clear argument, both orally and in writing, using prose, numbers and images

mathematics-specific actions

- effectively using mathematical representations, including symbols and graphs

In summary, since it is evident that there is no reasonable way to separate, nor should there be any interest in separating, the mathematics-specific and the general problem-solving aspects of the action dimension of mathematical knowledge, we feel it clearer and more helpful to parse the domain of mathematics as

Object (Number and Quantity, Shape and Space, Pattern and Function, Chance and Data, Arrangement), and
Action (including both mathematics-specific and general problem-solving actions)

ALGEBRA AND THE AUSTRALIAN CURRICULUM

Under the scope of the new Australian Curriculum, algebra instruction in mathematics falls under the content strand of 'Number and Algebra'. Within this strand, students are expected to become familiar with concepts such as number sense, recognising patterns, understanding variables and functions, solving inequalities and more.

This mathematics curriculum aims to ensure that students:

- are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens
- develop an increasingly sophisticated understanding of mathematical concepts and fluency with processes, and are able to pose and solve problems and reason in Number and Algebra, Measurement and Geometry, and Statistics and Probability
- recognise connections between the areas of mathematics and other disciplines and appreciate mathematics as an accessible and enjoyable discipline to study. (ACARA, 2013)

As algebra is included as an integral part of this new curriculum, it is important that students have access to the best schooling available, both through dedicated classroom instruction and through textbooks such as this. The Australian Curriculum expects students to have developed a strong understanding of algebra in the middle years, and have established a baseline for secondary school instruction.

In the middle years, specifically years 5–8 in this context, the Australian Curriculum concerns itself with six sub-strands under the 'Number and Algebra' strand: *Number and Place Value*; *Fractions and Decimals*; *Money and Financial Mathematics*; *Patterns and Algebra*; *Real Numbers*; and *Linear and Non-Linear Relationships*. References to these sub-strands will be made throughout the course of this book, in an effort to help teachers redefine their mathematics instruction under these guidelines.

The tasks featured throughout this book are presented with selected content descriptions from the Australian Curriculum, in an effort to help teachers bring their classroom instruction into alignment with this new framework.

THIS BOOK AND MIDDLE YEARS BALANCED ASSESSMENT IN MATHEMATICS

These tasks are assessments that help students further their understanding of important mathematical ideas and inform teachers and augment their instruction; in other words, assessments that teach as well as assess. Worthwhile assessment is not something students and teachers 'stop and do', but, rather, a way to further what they are already doing. These tasks can be used in classrooms to provide teachers, students, schools and parents with relevant information about mathematical learning.

In designing, testing and revising these tasks, we have worked closely with teachers in classroom settings to be sure that the abstractions of the academic world translate into workable, informative and useful classroom products – products that are coherent across year levels, across various content domains of mathematics, and across diverse interests of students and teachers. We have made every effort to balance the collection of tasks with respect to mathematical content covered, as well as the particular problem-solving skills necessary at various year levels. These assessments have been created with a unique design philosophy that gets to the heart of the question, "What is the structure of mathematics?"

WEIGHTING OF TASKS – WHAT ARE THE "INNARDS" OF PROBLEM SOLVING?

In order to approach the problem of designing assessment tasks, one must have a clear view of the kind of understanding and the skills that we wish to assess in our students, and the ways in which the tasks we design might elicit demonstrable evidence of those skills and understanding. In what follows we describe how our view of the subject of mathematics, its *objects* and its *actions*, informs the design of the tasks.

Each task is classified according to domain, that is, the mathematical objects that are prominent in the accomplishment of the task. Most of our tasks deal predominantly with a single sort of mathematical object, although some of the tasks deal with two or more such objects. Each task offers students an opportunity to demonstrate a variety of kinds of skill and understanding. Students must identify the most appropriate mathematical *object* for the task at hand and then decide what sequence of mathematical *actions* to carry out on the chosen *object*. Finally, students are asked to consider how effective their efforts were in addressing the task and to communicate to others what they did.

In order to score student performance on a task, we first analyse the task and decide on the nature of the demands that the task makes on the student. We consider the following four kinds of skill and understanding:

Modelling / Formulating: How well does the student understand the presenting statement and formulate or model the mathematical problem to be solved? Some tasks make minimal demands along these lines. For example, a problem that asks students to calculate the length of the hypotenuse of a right triangle, given the lengths of the two sides, does not make serious modelling demands. In this case, the student applies a known formula. On the other hand, the problem of how many 8-centimetre-diameter tennis balls can fit in a rectangular parallelepiped box that is 8 cm x 10 cm x 25 cm is a much more difficult modelling and formulating problem, even though its solution exercises the same Pythagorean muscles.

Transforming / Manipulating: How well does the student manipulate the mathematical formalism, that is, the algebraic expressions, the geometric shapes and so on, in which the problem is expressed? This may mean adding two numbers, dividing one fraction by another, making a geometric construction, solving an equation or inequality, plotting graphs, or finding the derivative of a function. Most tasks will make some demands along these lines. Indeed, most traditional mathematics assessment consists of problems whose demands are primarily of this sort.

Inferring/Drawing Conclusions: How well does the student apply the results of his or her mathematical actions to the problem situation that spawned the problem? How well does the student carry the information gained from one part of the problem-solving process to subsequent questions within the task? Traditional assessments often pose problems that make little demand of this sort.

Communicating: How well do students communicate to others what they have done in formulating the problem; manipulating the numbers, shapes, algebraic expressions; and drawing conclusions about the implications of their results?

Since we do not expect each task to make the same kinds of demands on students in each of the four skill/understanding areas, we assign a single-digit measure of the prominence of that skill/understanding in the problem according to the following scale of weighting codes. Note that these numbers are not measures of student performance, but are measures of the demands of the task for a given performance action. You might think of this as akin to rating the difficulty of a ski run (green circle, black diamond, etc) rather than the skill of the skiers who come down that gradient.

Weighting codes:

With respect to the skill / understanding being assessed in the task in question
0 = is not present at all
1 = is present in small measure
2 = is present in moderate measure and affects the solution
3 = has a prominent presence
4 = has a dominant presence

USING THE TASKS

Depending on your local needs and circumstances, these assessments can be used in a variety of ways.

- *Formative Assessment:* These tasks provide an opportunity for integrated, classroom-based formative assessment. The collection allows you to select tasks that are appropriate at particular points in the curriculum or that specifically address a mathematical action that students need help with. The collection of tasks is balanced in both content (mathematical *objects*) and process (mathematical *actions*) and is developmentally appropriate for the particular year level band. Using the tasks as formative assessment enables the teacher not only to adjust instructional strategy for the whole class, but also to pinpoint individual weaknesses.
- *Test Prep:* These tasks can be used as exemplars for open-response questions on high-stakes tests. They provide students with the opportunity to work on organising their mathematical thinking and to practise and refine their communication skills.
- *Support and Enrichment:* These tasks can serve as a support for a standards-based curriculum or as enrichment for an existing curriculum. They are designed to be used as a supplement to *any* standards-based curriculum. They can illuminate and advance students' thinking in new ways.
- *Diagnostic and Summative Assessment:* These tasks can be used as pre-test, post-test items for diagnostic purposes. They also may be used as summative assessment for specific topics or for additional information when assigning a mathematics grade.
- *Metacognitive Development:* These tasks can help students to become more reflective about their own mathematical thinking. They can provide a stimulus point for deeper classroom discourse.

AIDS TO CLASSROOM USE

At the beginning of each task you will find a teacher's guide that details

- the mathematical *object* category (some tasks fall into more than one category; the dominant object is marked with an **X and shown in bold**)

- the curriculum connection, noting topics in standard texts to which this assessment relates
- the process, or mathematical *action* weightings
- assumed mathematical background
- the core elements of performance
- specific directions for launching and conducting the task
- possible extensions
- materials (paper and pencil are assumed to be necessary for all tasks; in some cases, additional materials and calculators are indicated)

We have provided pre-activities in cases where a particular assessment may have an unfamiliar or unusual format, or when we want to suggest a variety of solution paths. By working through the pre-activity with the whole class, and answering any questions before assigning the main body of the task, the teacher can prepare students to be successful.

Depending on whether the task is being used as formative or summative assessment, the launch of a task will vary. In some cases it will be necessary only to distribute the task to students and let them read and work through it on their own. In other cases in may be more productive to have them work in pairs, but report back individually. If students are meeting this type of task for the first time, especially when tasks are being used primarily as learning tasks to enhance the curriculum, you may decide to work through them item by item, talking with the students and posing questions when things get 'stuck'. This type of informal assessment gives you the opportunity to observe what strategies students favour, what kinds of questions they ask, what they seem to understand and what they are struggling with, and what kinds of prompts get them 'unstuck'.

It is *extremely* important that you completely work through a task yourself before giving it to your students. Only in this way can you become familiar with the context and the mathematical demands. Only by having fully experienced the task yourself will you be able to anticipate what needs to be highlighted as you launch the task and where your students may run into difficulty. It is also imperative that you be aware of, and comfortable with, a wide range of possible solutions – in other words, there is often more than one 'right' answer or approach. When this is the case the rubrics will attempt to exhibit a range of possible solutions and approaches.

USING THE RUBRICS

Rubrics are a set of indicators or guidelines for giving scores to student work; they answer the question, "What do the varying degrees of mastery for this task look like?" The rubrics that accompany these assessments are based on the Core Elements of Performance that are identified for each task and may be used in a variety of ways. If the tasks are being used as formative assessment, students should be allowed to revise their work to meet, as closely as possible, the criteria for 'full competency'. If the tasks are being used for summative assessment, the partial- and full-competency descriptors can be restated as a four-level holistic rubric. Level 4 work meets all the descriptors for full competency; Levels 1–3 are arrived at by appropriately adjusting the descriptors for partial competency.

Teachers may need to translate scores from Balanced Assessment tasks to a letter- or number-grade system. While it may not be possible to preserve all four aspects of the mathematics actions, we *strongly urge* these scores be aggregated no further than two separate scores – one for skills (manipulating and transforming) and one for understanding (modelling, inferring). Collapsing the individual scores substantially reduces the utility of these materials to provide a mathematical profile of student understanding and to contribute to making informed decisions about students. Information on using these rubrics as part of a complete scoring system with accompanying software can be found in the document *MCAPS: Mathematical Content and Process Scoring* (Schwartz & Kenney, 1999).

