BUILDING PROPORTIONAL REASONING

Across Grades and Math Strands, K–8

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TEACHERS COLLEGE PRESS
TEACHERS COLLEGE | COLUMBIA UNIVERSITY
NEW YORK AND LONDON

NCTM
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1906 ASSOCIATION DRIVE | RESTON, VA 20191-1502
www.nctm.org

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INTRODUCTION

WHAT IS PROPORTIONAL THINKING?

Proportional thinking is based on recognizing and forming multiplicative comparisons between quantities. It involves thinking of numbers in relative terms rather than absolute terms. For example, when comparing 4 to 10, thinking of 10 as $2\frac{1}{2}$ fours rather than as 6 more than 4 is proportional thinking. Similarly, deciding that a price increase from \$2 to \$4 (a \$2 increase) is a more dramatic change than an increase from \$90 to \$100 (a \$10 increase), because the first price was doubled and the second was not nearly doubled, is thinking proportionally.

Another way to make sense of proportional thinking is to think of it as unitizing. Proportional thinking involves viewing one measurement (or amount) as so many units of another. It might be thinking of a set of 10 fingers as 2 units of 5 fingers, or it might be thinking of 1 m as 100 units of 1 cm.

Proportional thinking often requires transferring the use of a multiplicative relationship from one pair of numbers to another pair of numbers. For example, if you know that 3 identical items cost \$12 and want to know how many 6 will cost, you transfer the multiplicative relationship between 3 and 6 to a multiplicative relationship between 12 and some other number. Or you transfer the multiplicative relationship between 3 and 12 to the multiplicative relationship between 6 and another number. You are, in essence, determining equivalent ratios. In fact, in the National Council of Teachers of Mathematics (NCTM) resource *Developing Essential Understanding of Ratios, Proportions & Proportional Reasoning, Grades 6–8*, the big idea in proportional reasoning is described as a recognition that "when two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor" (Lobato, Ellis, Charles, & Zbiek, 2010, p. 11).

When Do We Use It?

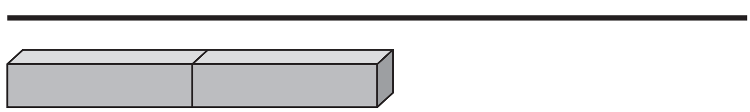
Proportional thinking, also often referred to as proportional reasoning, is used in everyday life in many ways. Just a few examples are listed on the next page:

- *Exchanging coins:* Every time you exchange quarters for nickels, or nickels for quarters, you change the unit of measure and, therefore, the number of units required. Because the exchange is always 5 nickels for 1 quarter, determining the number of nickels for so many quarters or vice versa is a use of proportional reasoning.
- *Changing measurement units:* Every time you change feet to inches or yards to feet or inches to yards, you change the unit of measure and, therefore, the number of units required. Changing a measurement from one unit to another is a use of proportional reasoning.
- *Calculating a best buy:* Every time you try to decide if so many gallons at one price is a better buy than a different number of gallons at a different price, you use proportional reasoning.
- *Planning a trip:* Every time you decide how many hours it will take for a trip based on an average speed, you use proportional reasoning.
- *Maps:* Every time you use a map with a given scale ratio to determine an actual distance, you use proportional reasoning.
- *Cooking:* Every time you adjust a recipe based on the number of people you want to feed or you figure out how many $\frac{1}{3}$ cup measures it would take to measure $\frac{1}{2}$ cup, you use proportional reasoning.

Where Is It in the Math Curriculum?

Although proportional reasoning is not formally mentioned as a topic in the Common Core math curriculum until 6th grade, its roots appear much earlier. Because proportional reasoning involves thinking of one number as a multiple of another (e.g., thinking of 6 as 2 threes or as 3 twos), it is applied as students begin to work in 3rd grade with simple multiplication and division. But proportional thinking actually begins even earlier than that.

For example, when students in earlier grades think about why they get to 50 quickly when they skip count by 10, but slowly when they skip count by 2, or when they think about why the whole line shown below is probably 4 rods long based on how far 2 rods extend along that line, they are building proportional reasoning.



Some work in place value can also be thought of in terms of proportional thinking. For example, realizing that 300 ones must be 3 hundreds since 100 ones is 1 hundred is an example of proportional thinking.

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As will be illustrated throughout this resource, work in a wide range of areas involves proportional reasoning: work in probability, work in creating and interpreting graphs with scales, all work with fractions, work with multiplication and division, some work with patterns, some work with solving equations, and some work involving linear relationships.

ESSENTIAL UNDERSTANDINGS RELATED TO PROPORTIONAL THINKING

The list below outlines some important understandings underlying proportional reasoning and pre-proportional reasoning that are useful and that should be addressed throughout the grades:

- It is often useful to think of one amount as so many units of another amount, for example, 1 dollar as 4 quarters, 7 days as 1 week, 20 eggs as $1\frac{2}{3}$ dozen eggs, etc.
- If you use a bigger unit, you need fewer of them to express a quantity. For example, it takes only 10 tens to make 100, but it takes 20 fives. Or, it only takes 1 yard to measure 3 feet. Or, 1 is 5 fifths, but only 2 halves.

  Another way to rephrase this idea is the following: Any amount can be a small amount of a big unit or a big amount of a small unit. For example, 5 is half of a 10, but only a quarter of a 20.
- If units are related, you can use that relationship to predict how many of one unit you will have if you know how many there are of the other. For example, since 4 quarters make 1 dollar, you can predict that 12 quarters makes 3 dollars.
- Any two numbers can be compared multiplicatively, even if one is less than the other. For example, just as 6 is two 3s, 3 is half of a 6.
- How far apart two numbers are additively is unrelated to how far apart they are multiplicatively. For example, the pair of numbers 3 and 6 and the pair of numbers 100 and 200 have the same multiplicative relationship, even though the first numbers are only 3 apart and the second are 100 apart.
- Using a fraction, a decimal, or a percent is a form of multiplicative comparison. For example, the reason $\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$ is because in each case the numerator is $\frac{2}{3}$ of the denominator, or the denominator is $\frac{3}{2}$ of the numerator. For example, the decimal 0.231 is a way to compare 231 to 1000. For example, the percent 42% is a way to compare 42 to 100.

These ideas emerge and re-emerge through this resource, through the grades, in the sections on underlying ideas, as well as in the sections listing Good Questions to Ask.

FOCUSING ON THE CCSSM STANDARDS FOR MATHEMATICAL PRACTICE

The CCSSM Standards for Mathematical Practice derive from the processes of the National Council of Teachers of Mathematics (NCTM, 2000) and the strands of mathematical proficiency from *Adding It Up* (National Research Council, 2001). The standards for mathematical practice describe the mathematical environment in which it is intended that the Common Core State Standards for Mathematics are learned. These standards for mathematical practice are meant to influence the instructional stance that teachers take when presenting tasks to help students grasp the content standards. The standards for mathematical practice are addressed in this resource both in the underlying ideas presented for each topic and in the types of Good Questions suggested.

Listed below are just a few examples of attention to each standard for mathematical practice in this resource.

1. ***Make sense of problems and persevere in solving them.*** Students in Grade 3 are encouraged to begin to think of multiplication in terms of a change of unit to help make sense of certain problems (page 37). Students in Grade 7 need to make sense of a problem that asks them to figure out which dimension change has the most effect on the surface area of a prism (page 90).

2. ***Reason abstractly and quantitatively.*** Proportional thinking is all about reasoning, so there are many reasoning opportunities presented in this resource. For example, in Grade 2, students reason about the relationship between 60 – 40 and 6 – 4 when they are stimulated to recognize the fact that they can describe the first situation as essentially the second one with a simple change of unit (from tens to ones) (page 28). Grade 5 students reason abstractly and quantitatively as they compare the growth rates of two patterns (page 54) and as they consider the value of digits in a place value situation (page 55). Using double number lines helps Grade 6 students make sense of how percents of numbers other than 100 relate to those numbers (pages 72–73).

3. ***Construct viable arguments and critique the reasoning of others.*** Based on work with counters, students in Kindergarten discover that it is not possible to create two equal groups when working with certain numbers of counters (pages 11–12)

and students in Grade 1 see that when halves of objects are bigger, so are the whole objects, or vice versa (page 22). Students in Grade 6 are asked to create an argument to predict why certain percent situations are not possible (page 74).

4. ***Model with mathematics.*** In Grade 1, students are modeling with mathematics when they measure half a distance and predict the whole distance (page 21). In Grade 7, students use mathematics to model probability situations (page 91). In Grade 8, students use dilations to create similar shapes (pages 98–99) and they use lines of good fit to model real-life situations (pages 102–103).

5. ***Use appropriate tools strategically.*** In Kindergarten, students use counters to explain principles (page 11). The 100-chart is a useful tool in Grade 1 to help students become familiar with multiples of 10 (page 18). Base-ten blocks are useful at many levels for many purposes, but one purpose is to help Grade 2 students see that counting larger subgroups is an efficient way to count (page 26). In Grade 4, using appropriate tools helps students understand various ways to compare two ratios or fractions (pages 46–47), and in Grade 7, 100-grids or double number lines help students calculate percents (page 81).

6. ***Attend to precision.*** Students in Grade 3 must attend to precision when they try to model a number as trains of another number, using Cuisenaire rods (pages 35–36). Students in Grade 7 consider precision when using strategies to get a sense of the size of π (page 87).

7. ***Look for and make use of structure.*** Students in Grade 1 start to recognize that when counting equal groups of objects, there are always two ways to count—either the number of groups or the number of items; the structure of every situation involving equal groups—multiplication—leads to this conclusion (pages 15–16). Students in Grade 3 might begin to notice that halving one number and doubling another leads to the same result because of what multiplication means (page 39). Students in Grade 7 begin to realize that if one variable is proportional to another, it is because the equation is of the form $y = mx$ and the graph is a line that goes through the origin (page 85).

8. ***Look for and express regularity in repeated reasoning.*** Students in Grade 2 might notice that there are two reasonable definitions for the notion of even number: either a number made up of a lot of twos or a number made up of two of the same whole number (page 23). Students in Grade 7 are expected to use tables of values involving proportional variables to see that 0 always matches 0, that 1 always matches r, where r is a unit rate of some sort, and that the y-values increase by r as the x-values increase by 1 (pages 83–85).

FOCUSING ON THE NCTM PRINCIPLES TO ACTIONS

Recently, the National Council of Teachers of Mathematics released *Principles to Actions: Ensuring Mathematical Success for All* (NCTM, 2014), articulating a vision for the conditions, structures, and policies that are critical to move mathematics education forward. Included are eight teaching practices, some of which focus on the tasks that teachers set and others on the pedagogical approach of the teacher. This resource directly supports many of those suggested practices.

For example, beyond the standards themselves, the underlying ideas articulated in this resource establish mathematical goals to focus learning and frequently use and connect mathematical representations. The tasks suggested as Good Questions are purposeful and promote reasoning and problem solving and meaningful discourse. The underlying ideas often require complex and non-algorithmic thinking.

FOCUSING ON THE CCSSM STANDARDS FOR MATHEMATICAL CONTENT

By its very organizational structure, this resource focuses on the mathematical content standards related to proportional thinking in the Common Core State Standards for Mathematics.