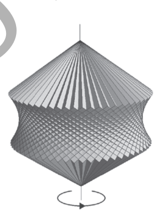
MATHEMATICAL *Imagining*

A ROUTINE FOR SECONDARY CLASSROOMS



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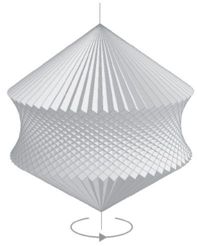
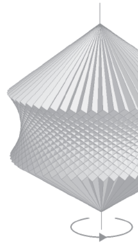


CONTENTS



PREFACE TO THE ENGLISH EDITION

I believe that the image is the great instrument of instruction. What a child gets out of any subject presented to him is simply the images which he himself forms with regard to it. . . . I believe that much of the time and attention now given to the preparation and presentation of lessons might be more wisely and profitably expended in training the child's power of imagery.

—John Dewey, *My Pedagogic Creed*

The power to imagine is usefully called upon explicitly and can be developed with practice. If imagination is not called upon in mathematics, then a powerful link to the emotions is neglected, and motivation-interest may suffer. If expression in multiple forms is not encouraged, then learners may form the mistaken impression that mathematics does not offer opportunities for creativity. If learners encounter a very limited range of images, and a very limited range of expressions, they are likely to form the erroneous impression that mathematics is a very limited domain of human experience.

—Sue Johnston-Wilder and John Mason, *Developing Thinking in Geometry*

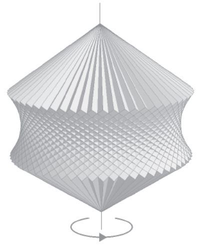
During my twenty-five years of teaching mathematics at the secondary-school level in Switzerland, I have seen firsthand the truth of these quotations—yes, we teachers can encourage our students to use their imaginations and mental images for learning, enjoying, and better understanding math. Mathematical Imagining was also the theme of my doctoral research, which helped to make explicit my tacit knowledge in action. In turn, the dissertation became the basis for this book, originally published in German in 2010 under the title *Mathematische Vorstellungsübungen im Unterricht*.

Since its publication, I have led several dozen professional development courses on using Mathematical Imagining in the classroom for teachers in German-speaking countries and also given many presentations about it. Meanwhile, interest in the topic has grown—and use of the Mathematical Imagining routine is spreading. So, it seems that my "child" has now matured enough to leave home, go out into the world, and meet teachers and students in other countries. To that end, I am pleased to be able to share the book with you now in this English version.

The term "exercises in imagining" was used previously by the mathematicians John Conway, Peter Doyle, Jane Gilman, and William Thurston. In their university summer workshop titled "Geometry and the Imagination," they promoted using imagery at the university math level, offering a series of exercises in Mathematical Imagining (Conway et al. 1991). But it is not only the occasional academic mathematician that considers imagery to be crucial for understanding. For example, some universities are now including work with imagining tasks in their mathematics teacher education programs. Furthermore, a number of preservice math teachers have incorporated Mathematical Imagining into their student teaching practice and written papers for seminars and bachelor's and master's theses that demonstrate the effects of using Mathematical Imagining in classrooms.

This handbook provides you with a collection of ideas and impulses to use in your lessons—ideas and lessons I have developed and refined in my own classroom, with my own students. It also demonstrates that imagining tasks can be implemented with certain nongeometrical content and can be used productively with your high school students so that they, too, can better learn, understand, and use the universal language of mathematics.

I wish you well in your use and exploration of the powerful and fascinating routine of Mathematical Imagining.



AN INVITATION INTO MY CLASSROOM

[Mathematical Imaginings] get me interested,
because they are more vivid than normal math teaching.

—Diana, high school student

My high school students stream into class late in the afternoon, many coming from gym. They drop their bags next to their chairs and put their math materials on their tables. I switch off the lights and ask students to prepare for an exercise in Mathematical Imagining. There is a perceptible change in energy—students are familiar with this weekly routine, and they push their books, notes, and pencils out of their way and take a few focused breaths. I invite students to sit comfortably and close their eyes. Some students sit up, hands relaxed in their laps, with their eyes closed. Others choose to fold their arms on their tables and drop their heads into the spaces they've made, to further block out light and distractions. They rest their foreheads on their forearms and listen, expectantly. I pick up my written exercise and begin reading it out loud, slowly and clearly.

> Imagine you are walking on a grassy field. You see three long strips of cloth lying in front of you in the shape of a *large triangle*. Stand in the middle of one of the strips of cloth with your nose and toes pointing into the triangle *in front of* you and stretch your arms out sideways to the left and right: they form a line *above* one strip—that is, parallel to and above one side of the triangle. . . .

I watch as students imagine. They are relaxed, concentrating, each one creating a personal field and triangle. I notice that, when I told them to imagine stretching their arms out sideways, a few students made barely perceptible movements in their shoulders or fluttered their fingertips a tiny bit, their physical bodies echoing what their *imaginary bodies* are doing inside their minds. I continue.

> Now begin moving *sideways* along the strip of cloth, keeping your nose and toes pointing *into* the triangle as you *sidestep* in the direction of your right arm, placing one foot *next to* the other, until you reach a corner where two strips of cloth meet. . . .

I wonder whether students are moving sideways as I intended when I wrote the exercise or whether some students have mentally turned their bodies and walked toward the first vertex. I remind myself to explore this question in our discussions later.

> Your *right* arm now extends beyond the figure, and your *left* arm is above the side of the triangle that you have just sidestepped along. Rotate slowly about your body's axis with your arms firmly outstretched so that your left arm begins to point *into* the figure. Keep rotating your body until your left arm has swept out the corner and arrived *above* the next strip of cloth, and then stop rotating.

I am curious how students are experiencing this moment, this chance to consider an interior angle of a triangle as a *dynamic rotation,* rather than a static measurement. They'll have two more chances to think about these turns.

> Begin sidestepping along the triangle's side, *this time* in the direction of your *left* arm, placing one foot *next to* the other, until you reach the second corner. Your *left* arm now extends beyond the triangle, and your *right* arm is above the side of the triangle that you have just walked along. Rotate again slowly so that your right arm initially points into the triangle, sweeps out the corner, and arrives parallel to and *above* the next side. . . .

I take a brief moment to conjure up my own imagined triangle—the one I created when I planned this exercise and can now access in an instant. Taking this moment to recollect helps me connect to the mathematics and also helps me ensure students have sufficient time to imagine their triangles in detail. After this pause, I bring them back to the start.

> Continue sidestepping and rotating like this until you arrive back where you started from. . . .

I am delighted to hear some audible gasps as students return to the start. I ask my first questions.

> How are you standing now?
>
> What has happened?

I give students time to contemplate these questions individually, quietly. Asking students, "How are you standing now?" encourages them to compare which way they were facing when they started and which way they are facing when they ended up. I expect some students are mentally going around the triangle again, either deepening or modifying their original journey. Students who may have turned in different directions at the vertices, rather than making all their turns in the same direction, have their first chance to adjust their image and move around the triangle again. The question "What has happened?" encourages students to think about their mental journey around the triangle as a whole, perhaps even from a new, bird's-eye perspective.

When I sense the class is ready, I ask the question that I always ask at the close of this routine:

> What did you imagine during this exercise in Mathematical Imagining?

I wait, again, for students to consider this question, before inviting to them to "come back to class." Students open their eyes, eager to discuss their imaginings with their peers. I invite students to talk about their imagining in pairs or threes. I walk around, listening as students talk with one another about which way they were facing at various points along their routes. To help their classmates understand their points of view, some students spontaneously make quick sketches or use gestures. Students are eager to share and are curious about one another's imaginings, so they listen intently. I see a few students close their eyes and revise their triangles in response to what they've heard from their classmates.

After a few minutes, I ask students a new question.

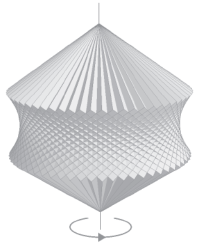
> What is the sum of the interior angles of a triangle?

Students now have a chance to reason together about why they have made a half-turn from beginning to end. Why a half-turn, exactly? And is it exactly a half-turn? Will it always be a half-turn, regardless of the triangle? These questions help lay the mathematical foundation for the interior angles theorem.

When conversation naturally begins to trail off, I ask students to take out their journals. Sometimes, I close our imaginings without journaling, but in this case, I decide to make time for students to capture their imagining before it starts to fade because their imaginings will be foundational to our upcoming work. For example, we'll eventually want to consider what happens when shuffling around a flat polygon with four vertices? Five? *n*? That discussion will be much richer if students can readily access their personal, mental triangles and modify them by adding sides and vertices. Therefore, I ask students to take a few minutes to respond to these prompts in writing before moving on to the rest of my lesson:

> Record the mental images—both pictures and actions—that you imagined during this exercise.
>
> Which of these images and actions were useful to you? Which got in your way? What revisions did you make?

Later, I'll look closely at their writing and sketches. I might select a few examples to share and have students discuss tomorrow. Or, I might choose to have all students leave their open journals on their tables and then walk around to read and comment on other students' work. Or, I might read them solely to inform my planning for upcoming lessons. Either way, this Mathematical Imagining gives my students a chance to create mathematics for themselves, and it helps me to see what is endlessly fascinating but not so easy to see: how my students think mathematically.



INTRODUCTION

An imagining task is a completely different world from a worksheet.
I can move there more freely.

—Peter, high school student

EXERCISES IN MATHEMATICAL IMAGINING

Mathematics can be an intellectual adventure, an adventure that takes place in our heads. It then thrives on the ability to imagine—objects and information are pictured in the "mind's eye" and further processed mentally. How can this imaginative aspect of mathematics be made to come alive and be implemented in the classroom?

Exercises in Mathematical Imagining cultivate the imaginative dimension of our discipline. One of the strengths of this new routine is that students engage with mathematical content directly and in a very personal way. This means that all students, not only the "high achievers," find a personal, inquiry-based approach to math. They learn to visualize math exercises more vividly and to work with their individual mental images. Students then find it easier to develop their own ways of solving a task. As we started to see in the introductory triangle example, working with imagining tasks in the classroom also gives you insight into your students' thought processes and how they go about tackling problems. Let's think about this abridged example of a different imagining task:

> Imagine a plastic cup lying on the floor. Give the cup a nudge so that it begins to roll. What does the path it takes look like?

This short text prompts your imagination, especially if instead of reading it out loud, you put yourself in the role of the listener. The words evoke mental images and actions that depict and elaborate the situation described. Thus, when you give the cup a mental push, you can "see" and perhaps even "sense" how it rolls forward. If your cup looks like a normal cup—a bit wider at the top than it is at the bottom—it will roll in a curve rather than in a straight line. You might have to push the cup more than once before it completes its trajectory and is finally back where it began. Is the trajectory in fact circular? With this conjecture, you are being mathematically active, and you are right in the middle of your own mathematical world.

This is precisely the effect of imagining tasks in the classroom. They open up a space in which students can engage with a mathematical situation, develop their own ideas and mental images, pose mathematical questions—and thus do mathematics in the best possible sense. Math begins in our heads!

INDIVIDUAL MENTAL IMAGERY IN THE CLASSROOM AND ITS VALUE FOR THE LEARNING PROCESS

The previous example encourages us to imagine a certain situation. However, it deliberately gives only a few pointers in order to leave room for individual, yet math-related mental images to take shape. Neither the exact shape nor the size and diameter of the cup are described, even if the wording *plastic cup* suggests a truncated right circular cone that can be held comfortably in one hand. The gradient and composition of the floor surface and the speed at which the cup rolls likewise remain unspecified. Only by exploring, elaborating, and thus acquiring an increasingly clear picture of a mathematical situation does it become mathematically accessible.

Once this foundation is present, it is possible for your students to perform actions and experiments in their minds, to formulate conjectures, and to be mathematically active. How they answer the question about "what the path looks like" (the shape of the trajectory) depends on how they imagine the rolling cup. Previous experience and basic mathematical knowledge allow them to make simple conjectures, such as "The cup rolls in a curve" or "The form taken by the trajectory depends on the shape and mass distribution of the cup." Whether it is a closed or even a circular "curve" is not yet clear. And neither is it clear which of the cup's geometrical and physical characteristics determine the exact form of the trajectory.